

A self-tuning mechanism in $(3 + p)$ d gravity-scalar theory

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ABSTRACT: We present a new type of self-tuning mechanism for $(3+p)$ d brane world models in the framework of gravity-scalar theory. This new type of self-tuning mechanism exhibits a remarkable feature. In the limit $g_s \rightarrow 0$, g_s being the string coupling, the geometry of bulk spacetime remains virtually unchanged by an introduction of the Standard Model(SM)-brane, and consequently it is virtually unaffected by quantum fluctuations of SM fields with support on the SM-brane. Such a feature can be obtained by introducing Neveu-Schwarz(NS)-brane as a background brane on which our SM-brane is to be set. Indeed, field equations naturally suggest the existence of the background NS-brane. Among the given such models, of the most interest is the case with $\Lambda = 0$, where Λ represents the bulk cosmological constant. This model contains a pair of coincident branes (of the SM- and the NS-branes), one of which is a codimension-2 brane placed at the origin of 2d transverse space ($\equiv \Sigma_2$), another a codimension-1 brane placed at the edge of Σ_2 . These two branes are (anti) T-duals of each other, and one of them may be identified as our SM-brane plus the background NS-brane. In the presence of the background NS-brane (and in the absence of Λ), the 2d transverse space Σ_2 becomes an orbifold R_2/Z_n with an appropriate deficit angle. But this is only possible if the $(3 + p)$ d Planck scale M_{3+p} and the string scale $M_s (\equiv 1/\sqrt{\alpha'})$ are of the same order, which accords with the hierarchy assumption [1–3] that the electroweak scale m_{EW} is the only short distance scale existing in nature.

KEYWORDS: p-branes, D-branes.

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1. Introduction

Inspired by the fact that the SM fields in string theory should be confined to the D-brane [1, 2], it was proposed that our universe may be a topological wall (3-brane) imbedded in a higher dimensional spacetime (bulk) [3]. This "brane world" scenario has drawn considerable attention over the last few years since it provides a mechanism for solving the longstanding puzzles such as the cosmological constant problem [4] or the hierarchy problem [2]. For instance, a 5d model was presented in [5] where the SM vacuum energy (or the brane tension) "warps" only the 5d bulk spacetime and does not affect the geometry of the brane itself, so the brane is kept flat regardless of the value of the brane tension. In this model, the desired TeV physical mass scale can be obtained from the fundamental Planck scale $\sim 10^{19}GeV$ through an exponential hierarchy generated by an exponential "warp" factor. Similar models also have been considered in 6d [6]. These models provide

a nice way to address the hierarchy problem, but they require fine-tuning between brane and bulk parameters in order to admit static solutions which are flat in the 4d brane world sector. Since these models are not free from the fine-tuning, a separate discussion would be necessary in order to meet the cosmological constant problem.

In this context, models without fine-tuning (or self-tuning models) have been proposed both in 5d [7–9] and 6d [10–12]. In particular, in [13] it was argued that the self-tuning of the cosmological constant is generic in 5d theories with no more than two branes coupled to a scalar field as well as gravity. The idea of self-tuning mechanism in 5d is that if in some cases the number of free parameters in the bulk solution is greater than the number of matching conditions (including orbifold boundary conditions), then one can find solutions where the brane tension can take any arbitrary value without changing geometry of the brane, and quantum corrections to the brane tension do not disturb the flatness of the brane. However, the presence of the bulk cosmological constant Λ leads to a reintroduction of the fine-tuning between Λ and the brane tension except for a particular case [8]. Also, 5d models generally involve a naked singularity from a finite proper distance from the brane [9, 13].

As mentioned above, self-tuning models also have been proposed in 6d. The self-tuning mechanism in 6d is different from that in 5d. Generally, 6d solutions have a desirable property that the brane tension does not affect the 4d effective cosmological constant; it only induces a deficit angle in the transverse space. The above 6d models are worth noticing in this respect. But still, certain kinds of fine-tunings are necessary in these models. For instance, a fine-tuning of bulk parameters is needed in [11] to obtain a sufficiently small value for the 4d cosmological constant, or a flux quantization causes a reintroduction of the fine-tuning between brane and bulk parameters in [12]. Besides this, there have been some claims [14] that these 6d models are not really the self-tuning models; they are indeed fine-tuning models. (Further conventional works in 6d can be found in [15]).

In this paper, along this line of studies, we will consider $(3+p)$ d brane world solutions (and corresponding brane world models) in the framework of gravity-scalar theory. The solutions show that the $(3+p)$ d gravity-scalar action admits a new type of self-tuning mechanism distinguished from the conventional ones. Namely, the geometry of bulk space-time remains virtually unaffected by the quantum fluctuations of SM fields with support on the SM-brane in the limit as the string coupling g_s goes to zero. Such a feature involves an introduction of a background NS-brane, whose existence is naturally suggested by field equations.

Historically, the theory with background NS-branes in the limit $g_s \rightarrow 0$ is not new. It is believed that in the limit $g_s \rightarrow 0$ NS5-branes of the type II string theory decouple from bulk modes (including gravity), and this decoupled theory of NS5-branes is known as "Little String Theory" (LST) [16]. For this reason the discussion of this paper may be partially regarded as an analogue of LST, and consequently the limit $g_s \rightarrow 0$ as an analogue of the decoupling limit of LST. Such an implication in LST is briefly discussed in section 6 and section 7.

Besides this, the self-tuning mechanism of this paper is naturally connected with the hierarchy problem. In the presence of the background NS-brane (and in the absence of Λ),

the 2d transverse space becomes an orbifold R_2/Z_n with a deficit angle $\delta = 2\pi(1 - \beta)$ with $\beta \sim (M_s/M_{3+p})^{1+p}$, where M_s is the string scale, while M_{3+p} the $(3 + p)$ d Planck scale. So in order that β becomes of order one, M_s and M_{3+p} must be of the same order, which naturally accords with the hierarchy conjecture that there may exist only one fundamental short distance scale (which is presumably the electroweak scale) in nature. We will see this in section 10.

2. Reduced action

We start with a $(3 + p)$ d action given by¹

$$I = \frac{1}{2\kappa^2} \int d^{3+p} X \sqrt{-G} \left[e^{-2\Phi} [R + 4(\nabla\Phi)^2] - e^{\alpha\Phi} \Lambda - \frac{1}{2} \sum_{i=1}^2 (\nabla\Psi_i)^2 \right], \quad (2.1)$$

where $2\kappa^2 = 1/2M_{3+p}^{1+p}$ in terms of the $(3 + p)$ d Planck scale M_{3+p} , and Φ is the $(3 + p)$ d dilaton. The cosmological constant term includes a factor $e^{\alpha\Phi}$, and where the case $\alpha = -2$ is of particular interest because it corresponds to the string theory. The action (2.1) also contains two scalar fields Ψ_1 and Ψ_2 , which will play an essential role in our self-tuning mechanism. To obtain equations of motion from (2.1) it is convenient to follow the lines of [17]. We take the $(3 + p)$ d metric to be of the form

$$ds_{3+p}^2 = e^{A(r)} d\hat{s}_3^2 + e^{B(r)} d\vec{x}_p^2 \quad (2.2)$$

where $d\vec{x}_p^2 = dx_1^2 + \dots + dx_p^2$, the line element of the p -brane, while

$$d\hat{s}_3^2 = -N^2(r) dt^2 + \frac{dr^2}{f^2(r)} + R^2(r) d\theta^2 \equiv \hat{g}_{ab} dy^a dy^b \quad (2.3)$$

represents a 3d subspace ($\equiv \Sigma_3$) with coordinates $y^a \equiv (t, r, \theta)$. In (2.3), θ is an angular variable with $0 \leq \theta \leq 2\pi$, and $(r, \theta) (\equiv \vec{r})$ are polar coordinates representing 2d transverse space ($\equiv \Sigma_2$). At this point we implicitly assume that the geometry of Σ_2 is locally R_2 around $\vec{r} = 0$.

The metric (2.2) is of the most general form with an $O(p)$ symmetry on the brane sector. In fact the field degrees of freedom of the metric (2.2) (together with (2.3)) are redundant. For instance the factor e^A could be absorbed into $d\hat{s}_3^2$, but it has been introduced for the later use. $f(r)$ is also a redundant degree of freedom associated with a coordinate transformation $r \rightarrow \tilde{r} = F(r)$. Now notice that the scale factors in (2.2) and (2.3) all depend only on r . We will also assume that the scalar fields Φ , Ψ_1 , and Ψ_2 are all functions of r alone (namely, we are considering rotationally symmetric p -brane solutions). Since the fields do not depend on the coordinates x_i along the p -brane, the $(3 + p)$ d action (2.1) can

¹Instead of the last term of (2.1), it is also possible to consider more generalized terms like $(-1/2) \sum_{i=1}^2 e^{\sigma_i} (\nabla\Psi_i)^2$, where σ_i is any dilatonic scalar other than Φ . The whole discussion of this paper still holds for this generalized Lagrangian as can be inferred from the argument of section 5.

be reduced to a 3d action defined on the subspace Σ_3 :

$$I_{\text{red}} = \frac{1}{2\kappa^2} \int d^3y \sqrt{-\det|\hat{g}_{ab}|} e^{-2\phi} \left[\hat{R} - e^A e^{(\alpha+2)\Phi} \Lambda - \frac{1}{4}(\partial A)^2 - \frac{p}{4}(\partial B)^2 + 4(\partial\phi)^2 - 2(\partial\phi)(\partial A) - \frac{1}{2}e^{2\Phi} \sum_{i=1}^2 (\partial\Psi_i)^2 \right], \quad (2.4)$$

where ϕ is the 3d effective dilaton defined by

$$\phi = \Phi - \frac{1}{4}A - \frac{p}{4}B, \quad (2.5)$$

and \hat{R} is the 3d Ricci-scalar obtained from \hat{g}_{ab} . The indices in (2.4) are all raised or lowered with \hat{g}_{ab} . If we choose (recall that e^A was a surplus degree of freedom)

$$A = 4\Phi - pB, \quad (2.6)$$

(2.4) reduces to

$$I_{\text{red}} = \frac{1}{2\kappa^2} \int d^3y \sqrt{-\det|\hat{g}_{ab}|} \left[\hat{R} - e^{(\alpha+6)\Phi - pB} \Lambda - 4(\partial\Phi)^2 + 2p(\partial\Phi)(\partial B) - \frac{p(p+1)}{4}(\partial B)^2 - \frac{1}{2}e^{2\Phi} \sum_{i=1}^2 (\partial\Psi_i)^2 \right], \quad (2.7)$$

where the 3d dilaton ϕ is absent and the kinetic term for \hat{g}_{ab} has the standard Einstein form without coupled to ϕ . So the first term of (2.7) is a standard Hilbert-Einstein action for the 3d gravity \hat{g}_{ab} , while the remaining terms constitute an effective matter action that gives a contribution to the energy-stress tensor T_{ab} of the 3d Einstein equations for \hat{g}_{ab} .

3. Codimension-2 brane and field equations with $\Psi_i = 0$

The bulk equations of motion off the brane can be obtained from the reduced action in (2.7). But in order to include the effect of the brane we need to introduce a brane action. In this section we start the discussion with an assumption that we have a codimension-2 brane at $\vec{r} = 0$. Also, we assume that the scalar fields Ψ_i are "turned off" ($\Psi_i = 0$) at this point. The action for a dilaton-coupled codimension-2 brane is most generally given by

$$I_{\text{cod-2}} = - \int d^{p+1}x \sqrt{-\det|g_{\mu\nu}|} V_p(\Phi), \quad (3.1)$$

where $V(\Phi)$ is an arbitrary functional of Φ , and $g_{\mu\nu}$ is a pullback of G_{AB} to the $(p+1)$ d brane world:

$$\begin{aligned} ds_4^2 &= G_{\mu\nu} dX^\mu dX^\nu \Big|_{\vec{r}=0} \\ &= -e^A N^2 dt^2 + e^B (dx_1^2 + \dots + dx_p^2) \Big|_{\vec{r}=0} \\ &\equiv g_{\mu\nu} dx^\mu dx^\nu. \end{aligned} \quad (3.2)$$

Upon using (2.6) and (3.2), the action I_{cod-2} can be converted into

$$I_{cod-2} = - \int_{\Sigma_3} d^3 y \sqrt{-det[\hat{g}_{ab}]} e^{2\Phi} V_p(\Phi) \delta^2(\vec{r}) , \quad (3.3)$$

where the 2d delta-function $\delta^2(\vec{r})$ has been normalized by

$$\int_{\Sigma_2} d^2 \vec{r} \sqrt{\hat{g}_2} \delta^2(\vec{r}) = 1 , \quad (3.4)$$

where \hat{g}_2 represents the determinant of the 2d metric defined on Σ_2 . As mentioned above we will begin our discussion with a codimension-2 brane placed at $\vec{r} = 0$. But later we will introduce a codimension-1 brane (in the case $\Lambda = 0$) at the edge of Σ_2 , which becomes a T-dual of the codimension-2 brane at $\vec{r} = 0$. Such a codimension-1 brane appears as a circle of certain radius, while the codimension-2 brane simply as a point on Σ_2 .

The field equations with a codimension-2 brane are now obtained from the total action $I_{total} \equiv I_{red} + I_{cod-2}$. In covariant form they are written as

(a) 3d Einstein equations:

$$\hat{R}_{ab} - \frac{1}{2} \hat{g}_{ab} \hat{R} = \kappa^2 (T_{ab} + t_{ab}) , \quad (a, b = t, r, \theta) \quad (3.5)$$

with

$$\begin{aligned} T_{ab} = & -\frac{1}{2\kappa^2} \hat{g}_{ab} e^{(\alpha+6)\Phi-pB} \Lambda + \frac{4}{\kappa^2} \left[(\partial_a \Phi)(\partial_b \Phi) - \frac{1}{2} \hat{g}_{ab} (\partial\Phi)^2 \right] \\ & - \frac{2p}{\kappa^2} \left[(\partial_a \Phi)(\partial_b B) - \frac{1}{2} \hat{g}_{ab} (\partial\Phi)(\partial B) \right] \\ & + \frac{p(p+1)}{4\kappa^2} \left[(\partial_a B)(\partial_b B) - \frac{1}{2} \hat{g}_{ab} (\partial B)^2 \right] , \end{aligned} \quad (3.6)$$

$$t_{ab} = -\hat{g}_{at} \hat{g}^{tt} \hat{g}_{tb} e^{2\Phi} V_p(\Phi) \delta^2(\vec{r}) , \quad (3.7)$$

where t_{ab} follows from the action I_{cod-2} in (3.3).

(b) Equations for Φ and B:

$$\square\Phi - \frac{p}{4} \square B - \frac{(\alpha+6)}{8} e^{(\alpha+6)\Phi-pB} \Lambda = \frac{\kappa^2}{2} e^{2\Phi} \left[V_p(\Phi) + \frac{1}{2} \frac{\partial V_p(\Phi)}{\partial\Phi} \right] \delta^2(\vec{r}) , \quad (3.8)$$

$$\frac{(p+1)}{4} \square B - \square\Phi + \frac{1}{2} e^{(\alpha+6)\Phi-pB} \Lambda = 0 . \quad (3.9)$$

Substituting (2.3) into the Einstein equations in (3.5) gives the following three independent equations of motion:

$$N(fR')' + \frac{1}{2} \frac{NR}{f} e^{(\alpha+6)\Phi-pB} \Lambda + NfR\mathcal{H} = -\kappa^2 \frac{NR}{f} e^{2\Phi} V_p(\Phi) \delta^2(\vec{r}) , \quad (3.10)$$

$$N'fR' + \frac{1}{2} \frac{NR}{f} e^{(\alpha+6)\Phi-pB} \Lambda - NfR\mathcal{H} = 0 , \quad (3.11)$$

$$(N'f)'R + \frac{1}{2} \frac{NR}{f} e^{(\alpha+6)\Phi-pB} \Lambda + NfR\mathcal{H} = 0 , \quad (3.12)$$

where $\mathcal{H} \equiv 2\Phi'^2 - p\Phi'B' + \frac{p(p+1)}{8}B'^2$, and the "prime" denotes the derivative with respect to r . Similarly, from (3.8) and (3.9) one obtains

$$(NfR\Phi')' - \left[\frac{(\alpha+2)(p+1)+4}{8} \right] \frac{NR}{f} e^{(\alpha+6)\Phi-pB} \Lambda \quad (3.13)$$

$$= \frac{(p+1)}{2} \kappa^2 \frac{NR}{f} e^{2\Phi} \left[V_p(\Phi) + \frac{1}{2} \frac{\partial V_p(\Phi)}{\partial \Phi} \right] \delta^2(\vec{r}),$$

$$(NfRB')' - \frac{(\alpha+2)}{2} \frac{NR}{f} e^{(\alpha+6)\Phi-pB} \Lambda = 2\kappa^2 \frac{NR}{f} e^{2\Phi} \left[V_p(\Phi) + \frac{1}{2} \frac{\partial V_p(\Phi)}{\partial \Phi} \right] \delta^2(\vec{r}). \quad (3.14)$$

Equations (3.10)–(3.14) constitute a complete set of linearly independent equations of motion. Now we set

$$N \equiv e^{-2\Phi + \frac{(p+1)}{2}B} h, \quad (3.15)$$

which (together with (2.6)) converts (2.2) into

$$ds_{3+p}^2 = e^{4\Phi-pB} \left(\frac{dr^2}{f^2} + R^2 d\theta^2 \right) + e^B \left(-h^2 dt^2 + d\vec{x}_p^2 \right). \quad (3.16)$$

For $h = f$ (and with $\Phi = (p-1)B/4$), (3.16) takes the usual form of the black brane. But in this paper we are interested in the case $h = 1$, i.e., the solution preserving $(p+1)$ d Poincaré invariance. By substituting (3.15) into (3.11)–(3.14) and comparing with one another, one obtains

$$\left(NfR \frac{h'}{h} \right)' = 0, \quad (3.17)$$

which shows that the $(p+1)$ d Poincaré invariance ($h = 1$) is automatically preserved. In what follows we will set

$$h = 1, \quad (3.18)$$

then

$$N \longrightarrow \xi \equiv e^{-2\Phi + \frac{(p+1)}{2}B}. \quad (3.19)$$

Due to (3.17), eqs. (3.10)–(3.14) are no longer linearly independent; only four of them are. Omitting (3.12), and choosing $f(r)$ as

$$f = \frac{r}{\xi R}, \quad (3.20)$$

one finds that the field equations reduce to the following set of four linearly independent equations:

$$\nabla^2 \ln R + \Lambda \psi = -\kappa^2 C_1 \delta^2(\vec{r}), \quad (3.21)$$

$$\nabla^2 \Phi - \left[\frac{(\alpha+2)(p+1)+4}{8} \right] \Lambda \psi = \frac{(p+1)}{2} \kappa^2 C_2 \delta^2(\vec{r}), \quad (3.22)$$

$$\nabla^2 B - \frac{(\alpha+2)}{2} \Lambda \psi = 2\kappa^2 C_2 \delta^2(\vec{r}), \quad (3.23)$$

$$-2 \left(r \frac{d \ln \xi}{dr} \right) \left(r \frac{d \ln R}{dr} \right) + 2 \left[2 \left(r \frac{d\Phi}{dr} \right)^2 - p \left(r \frac{d\Phi}{dr} \right) \left(r \frac{dB}{dr} \right) + \frac{p(p+1)}{8} \left(r \frac{dB}{dr} \right)^2 \right] = \Lambda \psi r^2, \quad (3.24)$$

where ∇^2 is the flat space Laplacian $\nabla^2 \equiv (1/r)(d/dr)(rd/dr)$ (so $\delta^2(\vec{r})$ is now normalized by $\int r dr d\theta \delta^2(\vec{r}) = 1$), while ψ and C_i are defined, respectively, by

$$\psi = \frac{\xi^2 R^2 e^{(\alpha+6)\Phi - pB}}{r^2} = \frac{R^2 e^{(\alpha+2)\Phi + B}}{r^2}, \quad (3.25)$$

and

$$C_1 = e^{(p+1)B/2} V_p(\Phi) \Big|_{\vec{r}=0}, \quad C_2 = e^{(p+1)B/2} \left[V_p(\Phi) + \frac{1}{2} \frac{\partial V_p(\Phi)}{\partial \Phi} \right] \Big|_{\vec{r}=0}. \quad (3.26)$$

By inspecting (3.25) together with (3.21)–(3.23) one finds that ψ must satisfy

$$\nabla^2 \ln \psi + m \Lambda \psi = \kappa^2 \left[2(C_2 - C_1) + \frac{(\alpha+2)(p+1)}{2} C_2 \right] \delta^2(\vec{r}) - 4\pi \delta^2(\vec{r}), \quad (3.27)$$

where

$$m \equiv 2 - \frac{(\alpha+2)[(\alpha+2)(p+1)+8]}{8}, \quad (3.28)$$

and similarly from (3.19), (3.22) and (3.23), we get

$$\nabla^2 \ln \xi + \Lambda \psi = 0. \quad (3.29)$$

Also in terms of ψ and ξ the metric (3.16) is rewritten as

$$ds_{3+p}^2 = e^{-(\alpha+2)\Phi} \psi \left(dr^2 + \frac{r^2}{\xi^2} d\theta^2 \right) + e^B (-dt^2 + d\vec{x}_p^2). \quad (3.30)$$

4. Solution to field equations

The solution to the differential equations in the previous section can be readily found. Using (3.27), one can show that the most general solution to the set of field equations (3.21)–(3.23) and (3.29) takes the form

$$R = i_R \psi^{k_R} R_0, \quad e^\Phi = i_\Phi \psi^{k_\Phi}, \quad e^B = i_B \psi^{k_B}, \quad \xi = i_\xi \psi^{k_\xi}, \quad (4.1)$$

where R_0 is a constant with length dimension one, and k_M ($M \equiv R, \Phi, B, \xi$) are given by

$$k_R = k_\xi = \frac{1}{m}, \quad k_\Phi = -\frac{[(\alpha+2)(p+1)+4]}{8m}, \quad k_B = -\frac{(\alpha+2)}{2m}. \quad (4.2)$$

Also i_M are defined by

$$\nabla^2 \ln i_M = 2\pi(\alpha_M + 2k_M) \delta^2(\vec{r}) \equiv 2\pi a_M \delta^2(\vec{r}), \quad (4.3)$$

where the constants α_M are given by

$$\alpha_R = -\frac{\kappa^2}{2\pi} \left[\left(1 - \frac{2}{m}\right) C_1 + \frac{2}{m} C_2 + \frac{(\alpha+2)(p+1)}{2m} C_2 \right], \quad (4.4)$$

$$\alpha_\Phi = -\frac{\kappa^2}{2\pi} \left[\frac{[(p+1)+1]}{m} C_2 - \frac{[(\alpha+2)(p+1)+4]}{4m} C_1 \right], \quad (4.5)$$

$$\alpha_B = -\frac{\kappa^2}{2\pi} \left[\frac{4}{m} C_2 - \frac{(\alpha+2)}{m} (C_1 + C_2) \right], \quad (4.6)$$

$$\alpha_\xi = -\frac{\kappa^2}{2\pi} \left[\frac{2}{m} (C_1 - C_2) - \frac{(\alpha+2)(p+1)}{2m} C_2 \right]. \quad (4.7)$$

By (4.7), (3.27) can be rewritten as

$$\nabla^2 \ln \psi + m\Lambda \psi = -2\pi (m\alpha_\xi + 2) \delta^2(\vec{r}) , \quad (4.8)$$

and in the case $\Lambda \neq 0$, the solution to (4.8) takes the form [18]

$$\psi(r) = \frac{\pm(8\gamma^2/m\Lambda)}{r^2[(r/r_0)^\gamma \pm (r/r_0)^{-\gamma}]^2} , \quad \left(\pm \equiv \frac{\Lambda}{|\Lambda|} \right) , \quad (4.9)$$

where r_0 is an arbitrary integration constant, but γ is given in terms of α_ξ :

$$\gamma = -\frac{m}{2}\alpha_\xi . \quad (4.10)$$

For $\Lambda = 0$, on the other hand, the solution to (4.8) can be written in the form

$$\psi(r) = \frac{b_0}{r^2(r/r_0)^{-2\gamma}} , \quad (4.11)$$

where b_0 is an arbitrary constant with length dimension two. In (4.9) (i.e., in the case $\Lambda \neq 0$), the constant γ must be positive. If $\gamma < 0$, ψ becomes $\psi \sim r^{-(2\gamma+2)}$ as $r \rightarrow 0$, and therefore $\nabla^2 \ln \psi \sim -2\pi (-m\alpha_\xi + 2) \delta^2(\vec{r})$, which does not agree with (4.8), i.e., ψ does not satisfy the boundary condition at $\vec{r} = 0$. In the case $\Lambda = 0$, however, γ (and consequently α_ξ) can be both positive and negative. Finally, the solution to (4.3) (for both $\Lambda \neq 0$ and $\Lambda = 0$) is simply

$$i_M(r) = \left(\frac{r}{\tilde{r}_0} \right)^{a_M} , \quad (a_M \equiv \alpha_M + 2k_M) , \quad (4.12)$$

where \tilde{r}_0 is an arbitrary constant.

Though the solution (4.1) satisfies (3.21)–(3.23) and (3.29), we still need for consistency to check whether it satisfies (3.24) either. Substituting (4.1) (together with (4.9) (or (4.11)) and (4.12)) into (3.24) gives two consistency conditions

$$\frac{2}{m}(\alpha_\xi + \alpha_R) + \frac{[(\alpha + 2) + 4]}{m}\alpha_\Phi - \frac{p}{m}\alpha_B = 0 , \quad (4.13)$$

and

$$-2\alpha_\xi\alpha_R + 4\alpha_\Phi^2 - 2p\alpha_\Phi\alpha_B + \frac{p(p+1)}{4}\alpha_B^2 = \frac{2}{m}\gamma^2 . \quad (4.14)$$

Since

$$\alpha_\xi = -2\alpha_\Phi + \frac{(p+1)}{2}\alpha_B \quad (4.15)$$

from the definition of ξ (and from (4.1) and (4.12)), the first consistency condition (4.13) can be rewritten as

$$2\alpha_R + \alpha_B + (\alpha + 2)\alpha_\Phi = 0 . \quad (4.16)$$

(4.16), however, is not a constraint. Using (4.4)–(4.6), one can show that (4.16) is identically satisfied from the definitions of α_M . The condition (4.14), however, imposes a restriction on α_M . Using (4.10), (4.15) and (4.16) one finds that (4.14) leads to a condition

$$l + \frac{(\alpha + 2)}{2} \equiv \hat{l} = 0 , \quad (4.17)$$

where l is defined by

$$l \equiv \frac{\alpha_B}{\alpha_\xi} . \tag{4.18}$$

In the case $\alpha = -2$ (or $m = 2$), (4.17) reduces to

$$l = 0 \quad \longleftrightarrow \quad \alpha_B = 0 , \tag{4.19}$$

which in turn implies $C_2 = 0$ (readers can check that $\hat{l} = 0$ in the case $\alpha \neq -2$ also leads to $C_2 = 0$), or equivalently

$$\left[V_p(\Phi) + \frac{1}{2} \frac{\partial V_p(\Phi)}{\partial \Phi} \right] \Big|_{\vec{r}=0} = 0 . \tag{4.20}$$

In the weakly coupled case of the string theory $V_p(\Phi)$ is expected to be a power series of the form

$$V_p(\Phi) = e^{\beta\Phi} \sum_{n=0}^{\infty} V_n e^{n\Phi} , \quad (V_n = \text{const.}) \tag{4.21}$$

(where $\beta = -1$ for the Dp -brane) when the quantum corrections to the brane tension are controlled by the dilaton. So if the brane at $\vec{r} = 0$ is a Dp -brane² with $V_p(\Phi)$ given by (4.21), V_n must be highly fine-tuned (for a given value of the string coupling $e^\Phi|_{\vec{r}=0}$) in order to satisfy (4.20), and this can hardly be accepted. Above all, D-brane does not satisfy (4.20) at the tree level.³ For the NS-brane, however, $V_p(\Phi)$ takes the form

$$V_p(\Phi) = V_0 e^{-2\Phi} , \quad (V_0 = \text{const.}) , \tag{4.22}$$

and (4.20) is immediately solved by (4.22) for arbitrary V_0 . This suggests that the codimension-2 brane at $\vec{r} = 0$ with $\Psi_i = 0$ would be an NS-NS type p -brane whose tension is given by (4.22). The type II and the heterotic string theories admit such a brane called NS5-brane. NS p -brane with $p < 5$ may then be regarded as a dimensionally reduced NS5-brane; for instance, NS3-brane can be taken to be a subsector of the NS5-brane wrapped on 2d compact space other than Σ_2 (see for instance [19]). This NS p -brane serves as a "background brane" on which the SM-brane (a Dp -brane with SM-fields living on it) is to be set.

5. Form invariant action

So far we have considered the case where Ψ_i are absent. Then we ended up with a result that the codimension-2 brane at $\vec{r} = 0$ should be an NS-brane. The action (2.1) with $\Psi_i = 0$ (and with $\Lambda = 0$ if necessary) precisely coincides with the low energy (closed) string action with zero NS-NS 2-form field. But the result is that this action does not seem to admit

²To be precise, the objects discussed in the present paper are not the real BPS-objects of the string theories, as the action does not include the terms for charges (i.e., n -form fields). The terms Dp -brane and NS-brane are used in analogy to string theory. However, we will proceed the discussion assuming as if they are BPS-branes, whose charges are neglected as is often the case with the usual brane world scenarios.

³The tree level form $V_p(\Phi) = V_0 e^{-\Phi}$ of the Dp -brane can satisfy (4.20) if we take $\Phi(\vec{r} = 0) = \infty$. But this is not consistent with the weakly coupled case in which e^Φ should not blow up on the D-brane.

a natural D-brane solution, which immediately gets us into trouble because SM fields live on a D-brane. In this section we will show that such a difficulty can be avoided if we allow nonzero Ψ_i in the action.

Turning back to the reduced action (2.4), one can show that the effect of introducing nonzero Ψ_i is to make a certain shift in Φ , A and B . Namely, in the case $\Lambda = 0$, the action with the field contents Φ , A , B and nonzero Ψ_i is equivalent to the action with the new fields $\tilde{\Phi}$, \tilde{A} , \tilde{B} and vanishing Ψ_i :

$$I(\Phi, A, B) \Big|_{\Psi_i \neq 0} = I(\tilde{\Phi}, \tilde{A}, \tilde{B}) \Big|_{\Psi_i = 0} , \quad (5.1)$$

where $\tilde{\Phi}$, \tilde{A} , \tilde{B} are given by

$$\tilde{\Phi} = \Phi + F_\Phi , \quad \tilde{A} = A + 4F_\Phi - pF_B , \quad \tilde{B} = B + F_B , \quad (5.2)$$

and F_Φ and F_B are related with Ψ_i by

$$\begin{aligned} \left[(4\partial F_\Phi - p\partial F_B) + (4\partial\Phi - p\partial B) \right]^2 - (4\partial\Phi - p\partial B)^2 &= 2e^{2\Phi} (\partial\Psi_1)^2 , \\ (\partial F_B + \partial B)^2 - (\partial B)^2 &= \frac{p}{2} e^{2\Phi} (\partial\Psi_2)^2 . \end{aligned} \quad (5.3)$$

In (5.3), $\Psi_1 = \Psi_2 = 0$ implies $F_\Phi = F_B = 0$ (we exclude the trivial case $F_\Phi, F_B = \text{constant}$), which means that F_Φ and F_B are field redefinitions of Ψ_1 and Ψ_2 . In addition to this, F_Φ and F_B also depend on e^Φ ; i.e., they are functionals of Ψ_1, Ψ_2 and e^Φ . (5.1) also holds for the case $\Lambda \neq 0$, but this time we have only one $\Psi_i (\equiv \Psi)$. That is, $\tilde{\Phi}$, \tilde{A} and \tilde{B} are given by

$$\tilde{\Phi} = \Phi + \frac{p}{4} F , \quad \tilde{A} = A - \frac{(\alpha+2)p}{4} F , \quad \tilde{B} = B + \frac{(\alpha+6)}{4} F , \quad (5.4)$$

and F is related with Ψ by

$$\left[\partial F + \frac{2}{(4-m)} (\partial B + \frac{\alpha+2}{2} \partial \ln \xi) \right]^2 - \left[\frac{2}{(4-m)} (\partial B + \frac{\alpha+2}{2} \partial \ln \xi) \right]^2 = \frac{4}{(4-m)p} e^{2\Phi} (\partial\Psi)^2 . \quad (5.5)$$

We see that $\Psi = 0$ implies $F = 0$ as before, and especially for the string theory ($\alpha = -2$), (5.5) reduces to

$$(\partial F + \partial B)^2 - (\partial B)^2 = \frac{2}{p} e^{2\Phi} (\partial\Psi)^2 . \quad (5.6)$$

Let us consider the field equations in the presence of nonzero Ψ_i . (5.2) (or (5.4)) shows that by appropriate field redefinitions Ψ_i can be absorbed into $\tilde{\Phi}$, \tilde{A} , \tilde{B} (in the forms of F_M ($M \equiv \Phi, B$) or F), and do not manifest themselves in the action. So the field equations (and consequently the solution) following from the action with $\Psi_i \neq 0$ are expected to take precisely the same form as those following from the action with $\Psi_i = 0$ except that (Φ, A, B) are replaced by $(\tilde{\Phi}, \tilde{A}, \tilde{B})$. This implies that for $\Psi_i \neq 0$, (4.20) should be replaced by

$$\left[V_p(\tilde{\Phi}) + \frac{1}{2} \frac{\partial V_p(\tilde{\Phi})}{\partial \tilde{\Phi}} \right] \Big|_{\tilde{r}=0} = 0 \quad \longleftrightarrow \quad \left[V_p(\tilde{\Phi}) + \frac{1}{2} \frac{\partial V_p(\tilde{\Phi})}{\partial \tilde{\Phi}} \right] \Big|_{\tilde{r}=0} \cong -\frac{1}{2} \frac{\partial F_\Phi}{\partial \tilde{\Phi}} \frac{\partial V_p(\tilde{\Phi})}{\partial F_\Phi} \Big|_{\tilde{r}=0} ,$$

which shows that (4.22) is not the correct solution for $V_p(\tilde{\Phi})$ anymore. Indeed from (5.7),

$$\begin{aligned} V_p(\tilde{\Phi}) &= e^{-2\tilde{\Phi}} V_0|_{\tilde{r}=0} \\ &= e^{-2\Phi} V_0(1 + f_\Phi)|_{\tilde{r}=0}, \end{aligned} \tag{5.8}$$

where f_Φ is defined by

$$F_\Phi = -\frac{1}{2} \ln(1 + f_\Phi). \tag{5.9}$$

So in order to admit a Dp -brane (SM-brane) in addition to the background NS-brane at $\tilde{r} = 0$, it is essential to allow f_Φ to have a nonzero value at $\tilde{r} = 0$. That is, if we define

$$f(0) \equiv V_0 e^{-\Phi} f_\Phi|_{\tilde{r}=0}, \tag{5.10}$$

(5.8) can be rewritten in the form

$$V_p(\tilde{\Phi}) = e^{-2\Phi} V_0 + e^{-\Phi} \sum_{n=0}^{\infty} V_n e^{n\Phi} + e^{-\Phi} \left[f(0) - \sum_{n=0}^{\infty} V_n e^{n\Phi} \right], \tag{5.11}$$

and in order that $V_p(\tilde{\Phi})$ includes the tensions of both D-brane and NS-brane, $f(0)$ must compensate the terms for the D-brane tension :

$$f(0) = \sum_{n=0}^{\infty} V_n e^{n\Phi}. \tag{5.12}$$

(5.10) is not a fine-tuning because $f(0)$ is arbitrary. Note that Ψ_i are subject to the equations of motion $\square\Psi_i = 0$ (with \hat{g}_{ab} in \square replaced by $e^{4\Phi}\hat{g}_{ab}$) which follow from (2.7). However, the boundary values of Ψ_i (or equivalently f_Φ at $\tilde{r} = 0$; see also footnote 5) can be chosen arbitrarily so as to satisfy (5.10) as we wish, and therefore (5.10) does not reduce to a fine-tuning. As a result, the codimension-2 brane at $\tilde{r} = 0$ is now a stack of an NS-brane and a D-brane in the presence of non-zero Ψ_i , and in this case the boundary values of Ψ_i are determined by the D-brane tension.

6. Geometry

Let us turn to the geometry of the $(3+p)$ d spacetime. In section 3, we found that $(3+p)$ d metric with a codimension-2 brane at $\tilde{r} = 0$ would take the form (3.30) with e^Φ , e^B , ξ and ψ given by (4.1), (4.9) (or (4.11)) and (4.12). Though we have used r -coordinates so far, it is also suggestive to introduce new coordinate systems.

In the case $\Lambda < 0$, we introduce a new variable χ defined by

$$\left(\frac{r}{r_0}\right)^{2|\gamma|} = \frac{(\chi - a)}{(\chi + a)}, \quad (a = \text{const.}). \tag{6.1}$$

If we take the constant a as

$$a^2 = \frac{|m|}{2} \frac{\alpha_\xi^2}{|\Lambda|r_0^2}, \tag{6.2}$$

then from (4.9) and (4.12), ψr^2 and i_M become, respectively,

$$\psi r^2 = r_0^2(\chi^2 - a^2), \quad (6.3)$$

and

$$i_M = d_M \left(\frac{\chi - a}{\chi + a} \right)^{2I_M}, \quad d_M \equiv \left(\frac{r_0}{\tilde{r}_0} \right)^{\alpha_M}, \quad (6.4)$$

where

$$I_M = -\frac{\epsilon}{m} \frac{\alpha_M}{\alpha_\xi}, \quad \left(\epsilon \equiv \frac{\gamma}{|\gamma|} \right). \quad (6.5)$$

The metric (3.30) now takes the form

$$\begin{aligned} ds^2 = & d_\Phi^{-(\alpha+2)} (\chi^2 - a^2)^{\frac{2}{m}-1-\frac{(\alpha+2)}{2m}} \left(\frac{\chi - a}{\chi + a} \right)^{-(\alpha+2)\hat{I}_\Phi} \\ & \times \left[\frac{2}{|m\Lambda|} \frac{d\chi^2}{(\chi^2 - a^2)} + \frac{r_0^2}{d_\xi^2} (\chi^2 - a^2)^{1-\frac{2}{m}} \left(\frac{\chi - a}{\chi + a} \right)^{-2\hat{I}_\xi} d\theta^2 \right] \\ & + d_B (\chi^2 - a^2)^{-\frac{(\alpha+2)}{2m}} \left(\frac{\chi - a}{\chi + a} \right)^{\hat{I}_B} (-dt^2 + d\vec{x}_p^2), \quad (\Lambda < 0), \end{aligned} \quad (6.6)$$

where \hat{I}_M is defined by

$$\hat{I}_M = -\frac{\epsilon}{m} \frac{\alpha_M}{\alpha_\xi}, \quad (6.7)$$

and therefore

$$\hat{I}_\xi = -\frac{\epsilon}{m}, \quad \hat{I}_B = -\frac{\epsilon}{m} l, \quad \hat{I}_\Phi = -\frac{\epsilon}{m} \left[\frac{(p+1)}{4} l - \frac{1}{2} \right]. \quad (6.8)$$

If $\Lambda > 0$, $(\chi - a)$ (and $(\chi^2 - a^2)$) in (6.6) must be replaced by $(a - \chi)$ (and $(a^2 - \chi^2)$). We further introduce η defined by

$$\chi = a \cosh \sqrt{|\Lambda|} \eta, \quad (6.9)$$

then (6.6) becomes

$$\begin{aligned} ds^2 = & d_\Phi^{-(\alpha+2)} a^{\frac{4}{m}-2-\frac{(\alpha+2)}{m}} \left(\sinh \sqrt{|\Lambda|} \eta \right)^{\frac{4}{m}-2-\frac{(\alpha+2)}{m}} \left(\tanh \frac{\sqrt{|\Lambda|}}{2} \eta \right)^{-2(\alpha+2)\hat{I}_\Phi} \\ & \times \left[\frac{2}{|m|} d\eta^2 + a^{2-\frac{4}{m}} \frac{r_0^2}{d_\xi^2} \left(\sinh \sqrt{|\Lambda|} \eta \right)^{2-\frac{4}{m}} \left(\tanh \frac{\sqrt{|\Lambda|}}{2} \eta \right)^{-4\hat{I}_\xi} d\theta^2 \right] \\ & + d_B a^{-\frac{(\alpha+2)}{m}} \left(\sinh \sqrt{|\Lambda|} \eta \right)^{-\frac{(\alpha+2)}{m}} \left(\tanh \frac{\sqrt{|\Lambda|}}{2} \eta \right)^{2\hat{I}_B} (-dt^2 + d\vec{x}_p^2), \quad (\Lambda < 0), \end{aligned} \quad (6.10)$$

and if $\Lambda > 0$, the hyperbolic functions in (6.10) should be replaced by trigonometric functions.

For $\Lambda = 0$, on the other hand, the new coordinate η is defined by

$$\frac{\eta}{\eta_0} \equiv \left(\frac{r}{r_0} \right)^{|\gamma|}, \quad (6.11)$$

and from (4.11) and (4.12) the metric (3.30) becomes

$$\begin{aligned}
 ds^2 = & d_\Phi^{-(\alpha+2)} \hat{b}_0^{\frac{2}{m}-1-\frac{(\alpha+2)}{2m}} \left(\frac{\eta}{\eta_0} \right)^{2\epsilon \left[\frac{2}{m}-1-\frac{(\alpha+2)}{2m} \right] - 2(\alpha+2)\hat{I}_\Phi} \\
 & \times \left[\frac{|m|}{2} \left(\frac{\eta}{\eta_0} \right)^{2(\epsilon-1)} \frac{d\eta^2}{\eta_0^2} + \hat{b}_0^{1-\frac{2}{m}} \frac{r_0^2}{d_\xi^2} \left(\frac{\eta}{\eta_0} \right)^{2\epsilon(1-\frac{2}{m})-4\hat{I}_\xi} d\theta^2 \right] \\
 & + d_B \hat{b}_0^{-\frac{(\alpha+2)}{2m}} \left(\frac{\eta}{\eta_0} \right)^{-\epsilon\frac{(\alpha+2)}{m}+2\hat{I}_B} (-dt^2 + d\vec{x}_p^2), \quad (\Lambda = 0), \quad (6.12)
 \end{aligned}$$

where we have set $b_0/r_0^2 \equiv \hat{b}_0$. Note that if we take

$$\hat{b}_0 = \frac{2}{m} |\gamma|^2 \frac{\eta_0^2}{r_0^2}, \quad (6.13)$$

(6.10) reduces to (6.12) in the limit $\Lambda \rightarrow 0$ provided that $\sqrt{|\Lambda|}$ is replaced with $2/\eta_0$.

The above metrics become greatly simplified when $\alpha = -2$ ($m = 2$). For $\alpha = -2$, (6.10) becomes

$$ds^2 = d\eta^2 + \frac{r_0^2}{d_\xi^2} \left(\tanh \frac{\sqrt{|\Lambda|}}{2} \eta \right)^{-4\hat{I}_\xi} d\theta^2 + d_B \left(\tanh \frac{\sqrt{|\Lambda|}}{2} \eta \right)^{2\hat{I}_B} (-dt^2 + d\vec{x}_p^2), \quad (\Lambda < 0), \quad (6.14)$$

where \hat{I}_ξ and \hat{I}_B are given by $\hat{I}_\xi = -\frac{1}{2}$ and $\hat{I}_B = -\frac{l}{2}$ (note that $\epsilon (\equiv \gamma/|\gamma|)$ must be positive in the case $\Lambda \neq 0$, see section 4). For $\Lambda = 0$, on the other hand, the metric is expected to be independent of α , as is obvious from (2.1). So in the case $\Lambda = 0$ we can choose any value for α (this means that (6.12) is invariant under the change of α), and in what follows we will always take $\alpha = -2$ ($m = 2$) in the case $\Lambda = 0$. For $\alpha = -2$, (6.12) reduces to

$$ds^2 = \left(\frac{\eta}{\eta_0} \right)^{2(\epsilon-1)} d\eta^2 + \frac{r_0^2}{d_\xi^2} \left(\frac{\eta}{\eta_0} \right)^{-4\hat{I}_\xi} d\theta^2 + d_B \left(\frac{\eta}{\eta_0} \right)^{2\hat{I}_B} (-dt^2 + d\vec{x}_p^2), \quad (\Lambda = 0), \quad (6.15)$$

where $\hat{I}_\xi = -\frac{\epsilon}{2}$ and $\hat{I}_B = -\frac{\epsilon l}{2}$.

Finally, the dilaton is given in the η -coordinates by

$$e^\Phi = \begin{cases} d_\Phi a^{2k_\Phi} \left(\sinh \sqrt{|\Lambda|} \eta \right)^{2k_\Phi} \left(\tanh \sqrt{|\Lambda|} \eta / 2 \right)^{2\hat{I}_\Phi(\epsilon=+1)}, & \text{for } \Lambda < 0, \\ d_\Phi \hat{b}_0^{k_\Phi} (\eta/\eta_0)^{2\epsilon k_\Phi + 2\hat{I}_\Phi}, & \text{for } \Lambda = 0, \end{cases} \quad (6.16)$$

which reduces for $\alpha = -2$ ($m = 2$) and $l = 0$ to

$$e^\Phi = \begin{cases} g_s / \cosh \sqrt{|\Lambda|} \eta / 2, & \text{for } \Lambda < 0, \\ g_s, & \text{for } \Lambda = 0, \end{cases} \quad (6.17)$$

where g_s is defined by

$$g_s = \begin{cases} d_\Phi a^{-\frac{1}{2}}/\sqrt{2} , & \text{for } \Lambda < 0 , \\ d_\Phi \hat{b}_0^{-\frac{1}{4}} , & \text{for } \Lambda = 0 . \end{cases} \quad (6.18)$$

The constant g_s in (6.17) plays the same role as the asymptotic value g_s of the theory with an ordinary NS5-brane. It can be taken to have any arbitrary desired value in (6.18) by choosing d_Φ properly. In our case we will take $g_s \rightarrow 0$, which corresponds to $d_\Phi \rightarrow 0$. The theory with NS-branes (or D-branes) in the limit $g_s \rightarrow 0$ is not new as mentioned in introduction. Such an idea can be found in the literatures on LST, where they consider the limit $g_s \rightarrow 0$ by which the bulk degrees of freedom decouple from the degrees of freedom of the brane, and one is left with physics on the brane. (Another example of using $g_s \rightarrow 0$ can be found in AdS/CFT correspondence where the gauge coupling $g_{YM}^2 (= g_s)$ goes to zero, while the rank N of the gauge group (or the number of D-branes) goes to infinity in such a way that Ng_{YM}^2 is held fixed.) In this paper we are essentially considering the same limit as those of these theories. Also, there is another important reason for considering the limit $g_s \rightarrow 0$. In our brane world models, taking $g_s \rightarrow 0$ naturally accords with the hierarchy conjecture as we will see in section 11. If we assume $M_s \sim TeV$ and $\rho_{\max} \sim TeV^{-1}$ (where M_s is the string scale: $M_s = 1/\sqrt{\alpha'}$, and ρ_{\max} is the size of Σ_2), g_s is estimated to be $g_s \sim 10^{-16}$, which, however, is just a realistic value of the decoupling limit $g_s \rightarrow 0$.

(6.17) shows that e^Φ becomes the constant g_s as we approach the background NS-brane ($\eta \rightarrow 0$). This contrasts with the case of the ordinary NS5-brane of the type IIA or type IIB string theory where e^Φ diverges in the vicinity of the NS5-brane. The reason why this happens is that the NS-brane discussed in the present paper is not the real BPS object of the string theory (see footnote 2). Had it been a BPS-brane with (magnetic) charge N , we would have had $e^{2\Phi} \sim -g_s^2 N \ln \eta$ (for $\Lambda = 0$) near $\eta = 0$, as can be inferred from the usual NS5-brane solution $e^{2\Phi} = g_s^2(1 + N\alpha'/\eta^2)$ which becomes $e^{2\Phi} \sim g_s^2 N\alpha'/\eta^2$ as $\eta \rightarrow 0$. So $e^{2\Phi}$ would have diverged as $\eta \rightarrow 0$, just as in the case of the usual NS5-brane. In general e^Φ diverges in the vicinity of the (BPS) NS-branes, and this divergence near singularity gives rise to an introduction of a new parameter ($\equiv g_{\text{lst}}$) in LST [20, 21], which serves as an effective string coupling on the D-branes stretched between N background NS5-branes. We will be back to this point later (see footnote 4).

By (6.18), one can estimate the constant r_0^2/d_ξ^2 appearing in the metrics (6.14) and (6.15). Using (6.2), (6.13), (6.18), and the relation $d_\xi = d_\Phi^{-2} d_B^2$ which follows from $i_\xi = i_\Phi^{-2} i_B^2$, one finds for $l = 0$ (and for $\alpha = -2$)

$$\frac{r_0^2}{d_\xi^2} = \begin{cases} 4g_s^4 \alpha_\xi^2 / |\Lambda| , & \text{for } \Lambda < 0 , \\ g_s^4 \alpha_\xi^2 \eta_0^2 , & \text{for } \Lambda = 0 , \end{cases} \quad (6.19)$$

where we have used the fact that $d_B = (r_0/\tilde{r}_0)^{l\alpha_\xi}$ when $\alpha = -2$ (see (6.4) and (4.18)), so $d_B = 1$ for $l = 0$. After all this, we observe that the $(3+p)$ d metrics essentially depend

on $V_p(\Phi)$ only through the constants l and α_ξ . In the next section we will show that the shifts in l and α_ξ due to quantum corrections to the D-brane tension are of an order g_s^2 , which implies that the change of the bulk geometry (including flat intrinsic geometry of the brane) due to quantum corrections to the D-brane tension is extremely suppressed in the limit $g_s \rightarrow 0$.

7. $V_p(\Phi)$ -independent bulk geometry

In the usual self-tuning brane world models, the intrinsic geometry of the brane is not affected by the brane tension $V_p(\Phi)$, but the geometry of bulk spacetime is always affected by (the change of) $V_p(\Phi)$. For instance in 6d (or codimension-2 brane world) models, the presence of a flat brane with the tension $V_p(\Phi)$ causes a deficit angle in the transverse dimensions the magnitude of which is proportional to $V_p(\Phi)$. Thus a change $\delta V_p(\Phi)$ in $V_p(\Phi)$ necessarily causes a corresponding change in the deficit angle, and this could lead to a failure of the self-tuning scheme. The $V_p(\Phi)$ -dependency of the bulk geometry, however, can be avoided by introducing background NS-brane; it is extremely suppressed in the limit $g_s \rightarrow 0$ due to the presence of the NS-brane. In this section we will show that the bulk geometry is really practically unaffected by the shift $\delta V_p(\Phi)$ in the limit $g_s \rightarrow 0$.

In section 5 we have seen that the solutions with $\Psi_i \neq 0$ (i.e., $\hat{l} \neq 0$) are basically self-tuning solutions. With $\hat{l} \neq 0$, the relation (4.18) can be expressed in terms of $V_p(\Phi)$ and $\partial V_p(\Phi)/\partial\Phi$ as

$$\alpha V_p(\Phi) + \frac{(\alpha - 2)}{4} \frac{\partial V_p(\Phi)}{\partial\Phi} = l \left[\frac{(\alpha + 2)(p + 1)}{4} V_p(\Phi) + \frac{[(\alpha + 2)(p + 1) + 4]}{8} \frac{\partial V_p(\Phi)}{\partial\Phi} \right], \quad (7.1)$$

where $V_p(\Phi)$ represents a sum of the tensions of the D-brane and the NS-brane; $V_p(\Phi) \equiv V_D(\Phi) + V_{NS}(\Phi)$, each of which is assumed to take the form

$$V_D(\Phi) = e^{-\Phi} \sum_{n=0}^{\infty} V_n^{(D)} e^{n\Phi}, \quad V_{NS}(\Phi) = e^{-2\Phi} V_0^{(NS)}. \quad (7.2)$$

Substituting (7.2) into (7.1) gives an ∞ -th order equation for $g_s (\equiv e^\Phi|_{\eta=0})$:

$$\sum_{n=0}^{\infty} c_n^{(l)} g_s^n = 0, \quad (7.3)$$

where

$$c_n^{(l)} = [k_1 + (n - 1)k_2] V_n^{(D)} g_s + [k_1 + (n - 2)k_2] V_0^{(NS)} \delta_{n0},$$

$$k_1 \equiv -\alpha + \frac{(\alpha + 2)(p + 1)}{4} l, \quad k_2 \equiv \frac{(l + 2)}{2} + \frac{(\alpha + 2)}{2} \left[\frac{(p + 1)}{4} l - \frac{1}{2} \right]. \quad (7.4)$$

Since α_M are combinations of $V_p(\Phi)$ and $dV_p(\Phi)/d\Phi$, the constant l would also depend on $V_p(\Phi)$ and $dV_p(\Phi)/d\Phi$, or equivalently on infinite numbers of $V_n^{(D)}$ and $V_0^{(NS)}$. Thus solving (7.3) for l (assuming that it can be solved) gives l in terms of $V_n^{(D)}$ and $V_0^{(NS)}$ for a given value of g_s .

Though we wish to solve (7.3) for l for a given value of g_s , it is convenient to begin with an assumption that (7.3) is an n -th order (∞ -th order in fact) equation for g_s . To solve (7.3), rewrite it as

$$\begin{aligned}
 - \left\{ l + \frac{(\alpha + 2)}{2} \right\} V_0^{(NS)} + \left\{ \left(1 - \frac{l}{2} \right) + (\alpha + 2) \left[\frac{(p+1)}{8} l - \frac{3}{4} \right] \right\} V_0^{(D)} g_s \\
 + \left\{ 2 + (\alpha + 2) \left[\frac{(p+1)}{4} l - 1 \right] \right\} V_1^{(D)} g_s^2 + \dots = 0 .
 \end{aligned} \tag{7.5}$$

(7.5) has a peculiar form. The n -th order coefficients $V_n^{(D)}$ appear as an $(n+1)$ -th order coefficients in (7.5). Equation (7.5) admits real solutions representing the limit $g_s \rightarrow 0$ for a certain value of l . Neglecting higher-order terms for a moment (and assuming that l is not infinitely large), one finds that the first two terms in (7.5) can cancel each other when $\hat{l} V_0^{(NS)}$ is much smaller than $V_0^{(D)}$. To be precise, (7.5) requires

$$\hat{l} \sim \left(V_0^{(D)} / V_0^{(NS)} \right) g_s , \tag{7.6}$$

provided the higher order terms are neglected. This is important. (7.6) implies that \hat{l} should be as small as g_s if $V_0^{(NS)}$ is of the same order as $V_0^{(D)}$; i.e., $\hat{l} \sim g_s$ if $V_0^{(NS)} \sim V_0^{(D)}$. (In string theory, $V_0^{(NS)}$ and $V_0^{(D)}$ both take the same value $V_0^{(D,NS)} \sim 1/\alpha'^3$ for $p = 5$.) So if $g_s \rightarrow 0$, \hat{I}_B and similarly \hat{I}_Φ are practically not different from their values with $\hat{l} = 0$. In the previous section we have observed that the $(3+p)$ d metrics depend on $V_p(\Phi)$ only through the constants l (i.e., \hat{l}) and α_ξ . Apart from α_ξ , this implies that the bulk geometry is practically unchanged by an addition of the SM-brane (the D-brane) to the background NS-brane. Note that Σ_2 with an NS-brane alone corresponds to $\hat{l} = 0$ ($\Psi_i = 0$), while Σ_2 with both SM- and NS-branes corresponds to $\hat{l} \neq 0$ ($\Psi_i \neq 0$).

Once \hat{l} is determined at the tree-level for the given values of $V_0^{(D,NS)}$, the effect of the higher order terms can be obtained by adding δl to \hat{l} , where $\delta l (= \delta \hat{l})$ is the shift in l due to quantum corrections to the tension of the SM-brane. As is obvious from (7.5), δl is proportional to g_s^2 ; i.e.,

$$\delta l \sim \left(V_1^{(D)} / V_0^{(NS)} \right) g_s^2 . \tag{7.7}$$

To estimate the magnitude of δl , first consider the case where the gauge coupling is simply given by $g_{YM}^2 \sim g_s \alpha'^{(p-3)/2}$, and $V_1^{(D)} / V_0^{(NS)}$ is of order the unity. In this case δl is simply

$$\delta l \sim g_{YM}^4 \alpha'^{-(p-3)} , \tag{7.8}$$

i.e., the shift in l due to quantum corrections to the brane tension is suppressed with the factor g_{YM}^4 .

There is a different way of viewing (7.7). Suppose that $V_D(\Phi)$ is written in the form

$$V_D(\Phi) = \frac{V_0^{(D)}}{g_s} \left(1 + \sum_{n=1}^{\infty} m_n g^n \right) , \tag{7.9}$$

where the terms with $n \geq 1$ describe the quantum corrections to the brane tension due to SM-fields living on the D-brane. The coefficients m_n are dimensionless, and $m_n \sim O(1)$

provided n is not very large. The constant g is a dimensionless (effective) coupling defined on the D-brane, and g_{YM} is now given by $g_{YM}^2 \sim g\alpha'^{(p-3)/2}$, while g_s is taken to be $g_s \rightarrow 0$. The expansion (7.9) may be applied to both cases where the quantum corrections are dilaton dependent, and where the quantum corrections are dilaton independent. In the former case the coupling g is given in terms of g_s (see (7.10)), and it becomes an analogue of g_{lst} of LST⁴ where the effective coupling on the D-brane is given by g_{lst} , while the coupling to the bulk modes behaves as g_s (i.e., while $V_D(\Phi)$ is given by $V_D(\Phi) \sim 1/\alpha'^{(p+1)/2}g_s$ at the tree level, its quantum corrections should be expanded in the SM coupling g_{YM} , or equivalently g_{lst}). Comparing (7.9) with (7.2) one finds that

$$g = \left(\frac{1}{m_n} \frac{V_n^{(D)}}{V_0^{(D)}} \right)^{1/n} g_s, \tag{7.10}$$

and (7.7) becomes

$$\delta l \sim g_{YM}^2 g_s \alpha'^{-(p-3)/2}. \tag{7.11}$$

So the shift δl is suppressed with the factor $\sim g_{YM}^2 g_s$ this time. Finally, in the case where the quantum corrections are dilaton independent, g simply represents the SM coupling g_{YM} (through the equation $g_{YM}^2 \sim g\alpha'^{(p-3)/2}$) which is now independent of g_s . But in this case too, one can show that δl is also given by (7.11). So in any case, the change of the bulk geometry due to quantum corrections is extremely suppressed in the limit $g_s \rightarrow 0$.

In the above discussion we have implicitly assumed that (7.3) can be solved for g_s . But in reality, it is impossible to solve the ∞ -th order equation, and we are only allowed to solve the n -th order equation with finite n (perhaps for $n \leq 4$). For instance if $n = 2$, the equation takes the form

$$\hat{c}_2^{(l)} g_s^2 + \hat{c}_1^{(l)} g_s + \hat{c}_0^{(l)} = 0, \tag{7.12}$$

where $\hat{c}_n^{(l)}$ do not include g_s . (7.12) admits real solutions for g_s as long as the condition

$$\left(\hat{c}_1^{(l)} \right)^2 - 4\hat{c}_2^{(l)}\hat{c}_0^{(l)} \geq 0 \tag{7.13}$$

is met. For (7.5), and for $\alpha = -2$ for simplicity, the condition (7.13) reads

$$\left(1 - \frac{l}{2} \right)^2 \left(V_0^{(D)} \right)^2 + 8lV_0^{(NS)}V_1^{(D)} \geq 0, \tag{7.14}$$

⁴As mentioned in section 6, LST admits an effective coupling g_{lst} on the D-brane [20]. g_{lst} is defined by $g_{\text{lst}} \sim g_s/LM_s$, where L represents the separation of the N NS-branes which are distributed uniformly on a transverse circle in moduli space. In the double scaling limit $g_s, L \rightarrow 0$ with g_s/L held fixed, g_{lst} takes a certain finite value and it plays a role of the effective string coupling on the D-branes stretched between N NS-branes. So in order to define g_{lst} , it is necessary to have a configuration that there are N (BPS) NS-branes sitting around the singularity. In the present paper, however, we are only considering a brane world scenario which uses LST, rather than being LST itself, only as a partial analogue of the theory, and we simply assume (without extending to the configuration with N (BPS) NS-branes) that an effective coupling g (which is dimensionless, and associated with g_{YM} by the equation $g_{YM}^2 \sim g\alpha'^{(p-3)/2}$) is intrinsically defined on the D-brane (SM-brane) as an analogue of g_{lst} of LST.

and the solutions becomes

$$g_s = \left[- \left(1 - \frac{l}{2} \right) V_0^{(D)} \pm \sqrt{\left(1 - \frac{l}{2} \right)^2 \left(V_0^{(D)} \right)^2 + 8l V_0^{(NS)} V_1^{(D)}} \right] / 4V_1^{(D)}. \quad (7.15)$$

(7.14) does not lead to a fine-tuning of $V_n^{(NS,D)}$. It only restricts the ranges of $V_n^{(NS,D)}$. Also one can check that one of the solutions in (7.15) reduces to $g_s \sim lV_0^{(NS)}/V_0^{(D)}$ for $l \rightarrow 0$, which is just the one that we have obtained in (7.6). The other solution in (7.15) reduces (upon using $V_0^{(D)} \sim V_0^{(NS)}$) to $g_s \sim -V_0^{(D)}/2V_1^{(D)}$ for $l \rightarrow 0$, which may correspond to the strongly coupled case for either $V_0^{(D)}/V_1^{(D)} \sim O(1)$ or $V_0^{(D)}/V_1^{(D)} \sim g_s/g$ (see (7.10)), and should perhaps be discarded in the framework using perturbation.

So far we have concentrated on the constant l . But the $(3+p)$ d metrics also depend on α_ξ (as well as l) as observed in section 6. But still, one can show that their geometries are virtually unchanged by an addition of the SM-brane though the effect of $\delta\alpha_\xi$ is taken into account. In order to see this, define a constant k as

$$k = \frac{\alpha_\xi(V_D(\Phi) + V_{NS}(\Phi)) - \alpha_\xi(V_{NS}(\Phi))}{\alpha_\xi(V_{NS}(\Phi))} = \frac{\delta\alpha_\xi}{\alpha_\xi(V_{NS}(\Phi))}, \quad (7.16)$$

where $\alpha_\xi(V_D(\Phi) + V_{NS}(\Phi))$ represents the value of α_ξ when the brane at $\eta = 0$ is a coincident brane of the SM- and the background NS-brane, while $\alpha_\xi(V_{NS}(\Phi))$ the value of α_ξ when the brane at $\eta = 0$ is simply an NS-brane. From (4.7), (7.2) and (7.16), one obtains

$$kV_0^{(NS)} + \frac{1}{2} \left[\frac{(\alpha + 2)(p + 1)}{4} - 1 \right] V_0^{(D)} g_s + \frac{(\alpha + 2)(p + 1)}{4} V_1^{(D)} g_s^2 + \dots = 0, \quad (7.17)$$

and the same analysis made for (7.5) can be applied just as it is to this case too. One finds $k \sim g_s$, implying (together with $\hat{l} \sim g_s$) that the $(3+p)$ d geometries are virtually unaffected by an addition of the SM-brane in the limit $g_s \rightarrow 0$. Similarly, the shift in k becomes $\delta k \sim \frac{1}{g_s^2}$ (In the case $\alpha = -2$, it is even smaller; i.e., $\delta k \sim g_s^3$.) as before, so the change of the $(3+p)$ d geometry due to quantum corrections is extremely suppressed in the limit $g_s \rightarrow 0$.

8. Codimension-1 brane as a T-dual of codimension-2 brane

In section 4, we found that solution to the set of field equations takes the form of (4.1) with $\psi(r)$ given by (4.9) or (4.11) according to whether $\Lambda \neq 0$ or $\Lambda = 0$. In the case $\Lambda \neq 0$, (4.9) is valid only when γ is positive. But in the case $\Lambda = 0$, (4.11) is valid for both positive and negative γ . So far we have assumed that the brane at $r = 0$ (or $\eta = 0$) is a codimension-2 brane. But codimension-2 brane can exist only when Σ_2 is closed at $r = 0$. In some cases Σ_2 fails to be closed at $r = 0$; rather, it spreads out (i.e., $\sqrt{g_{\theta\theta}}$ diverges) there. Whether Σ_2 is closed at $r = 0$ or not entirely depends on the signature of γ . When $\Lambda \neq 0$, Σ_2 is always closed at $r = 0$ ($\eta = 0$) because γ must be positive in the case $\Lambda \neq 0$, and $2 - (4/m) - 4\hat{I}_\xi = 2 > 0$ in (6.10) for positive γ . When $\Lambda = 0$, on the other hand, (6.15) shows that while Σ_2 is closed at $r = 0$ ($\eta = 0$) if $\epsilon > 0$ ($r > 0$), it spreads out as $r \rightarrow 0$

($\eta \rightarrow 0$) if $\epsilon < 0$ ($r < 0$). So the brane at $r = 0$ is a codimension-2 brane in the case $\epsilon > 0$, while it is necessarily a codimension-1 brane in the case $\epsilon < 0$. In the following discussion, we will identify these branes as certain limits of the type II codimension-1 brane introduced in appendix.

Once these branes (of the case $\Lambda = 0$) are identified with the type II codimension-1 branes, we observe that they are T-duals of each other provided that they have the same mass. In order to see this, return to the metric (6.15). For $\epsilon > 0$, (6.15) becomes

$$ds_{\epsilon>0}^2 = d\eta^2 + \frac{r_0^2}{d_\xi^2} \left(\frac{\eta}{\eta_0}\right)^2 d\theta^2 + d_B \left(\frac{\eta_0}{\eta}\right)^l (-dt^2 + d\vec{x}_p^2) , \quad (8.1)$$

while for $\epsilon < 0$,

$$ds_{\epsilon<0}^2 = \left(\frac{\eta_0}{\eta}\right)^4 d\eta^2 + \frac{r_0^2}{d_\xi^2} \left(\frac{\eta_0}{\eta}\right)^2 d\theta^2 + d_B \left(\frac{\eta}{\eta_0}\right)^l (-dt^2 + d\vec{x}_p^2) . \quad (8.2)$$

These two metrics are related to each other by a duality transformation. We see that one of them is converted into another by a transformation

$$\eta \rightarrow \tilde{\eta} = \frac{\eta_0^2}{\eta} , \quad (8.3)$$

and if we identify η_0 with the string length $\sqrt{\alpha'}$, (8.3) becomes a (closed string) T-duality transformation of the string theory. Codimension-1 branes can be obtained from (8.1) and (8.2) by fixing η to a constant ; i.e., $\eta = \eta_c$, where η_c represents the position (or the radius) of the branes in the η -coordinates. If these two codimension-1 branes have the same mass, they are T-duals of each other because they are related by the duality transformation (8.3), and the total mass of the brane is conserved under duality transformation. Let us identify these branes as the type II codimension-1 branes, and take $\eta_c \rightarrow 0$. The codimension-1 brane of the case $\epsilon > 0$ then shrinks by $\eta_c \rightarrow 0$ to a point to become a codimension-2 brane, while the other one of the case $\epsilon < 0$ still remains to be a codimension-1 brane. But still, these two branes are related by (8.3), and they have the same mass because the total mass of the type II codimension-1 brane is preserved under the variation of η_c . So they are T-duals of each other.

Let us consider the codimension-1 brane of the case $\epsilon < 0$. We assume that this codimension-1 brane is located at $r = 0$ for the moment. Since this brane exists only when $\Lambda = 0$, it is described by

$$\psi r^2 = \frac{b_0}{(|r|/r_0)^{-2\gamma}} , \quad (8.4)$$

and

$$i_M(r) = d_M \left(\frac{|r|}{r_0}\right)^{a_M} . \quad (8.5)$$

One can check that (8.4) and (8.5) satisfy the field equations (A.13) with $\Lambda = 0$, and (A.14), respectively. (8.4) and (8.5) show that there is a reflection symmetry about the brane at $r = 0$. We take this codimension-1 brane as an orbifold fixed line; i.e., we identify every

point of the region $r < 0$ with the corresponding point of the region $r > 0$, and then take the region $r > 0$ as a fundamental domain. Note that extra factor two has been multiplied on each α_M in (A.13) and (A.14). It reflects the fact that the orbifold fixed line at $r = 0$ is an overlap of two codimension-1 branes each of which belongs to the corresponding regions $r < 0$ and $r > 0$. So the tension $V_{p+1}(\Phi)$ (and consequently α_M) must be doubled.

So far we have assumed that the type II codimension-1 brane is placed at $r = 0$. However, we want the brane at $r = 0$ to be a codimension-2 brane because we want to set our SM-brane (a codimension-2 brane) at $\vec{r} = 0$. Thus in the followings, the type II codimension-1 brane will be moved to $r = r_m$ (or $\eta = \eta_m$ in the η -coordinates), and it will serve as a T-dual of the codimension-2 brane at $\vec{r} = 0$ (see the case III of section 10). So in the present paper we always consider the case where Σ_2 is closed at $r = 0$.

9. Matching conditions

According to the metrics in section 6, two extra dimensions of Σ_2 form an infinite volume space, which may need to be compactified anyhow. In order to avoid such noncompact extra dimensions, we introduce a codimension-1 brane (\equiv brane B) at a finite distance from the codimension-2 brane at $\vec{r} = 0$. Because this codimension-1 brane is expected to be an ordinary codimension-1 brane, we begin with the field equations for the type I codimension-1 brane introduced in appendix. We assume that the brane B is placed at $r = r_B$.

The field equations (A.3)–(A.5) require the fields to satisfy the matching conditions

$$r \frac{d \ln R_{II}}{dr} \Big|_{r=r_B} - r \frac{d \ln R_I}{dr} \Big|_{r=r_B} = -\kappa^2 C_1^{(p+1)}, \quad (9.1)$$

$$r \frac{d \Phi_{II}}{dr} \Big|_{r=r_B} - r \frac{d \Phi_I}{dr} \Big|_{r=r_B} = \frac{(p+1)}{2} \kappa^2 C_2^{(p+1)} + \frac{1}{2} \kappa^2 C_1^{(p+1)}, \quad (9.2)$$

$$r \frac{dB_{II}}{dr} \Big|_{r=r_B} - r \frac{dB_I}{dr} \Big|_{r=r_B} = 2\kappa^2 C_2^{(p+1)}, \quad (9.3)$$

in addition to

$$R_I \Big|_{r=r_B} = R_{II} \Big|_{r=r_B}, \quad e^{\Phi_I} \Big|_{r=r_B} = e^{\Phi_{II}} \Big|_{r=r_B}, \quad e^{B_I} \Big|_{r=r_B} = e^{B_{II}} \Big|_{r=r_B}, \quad (9.4)$$

where the indices I and II represent the regions $r < r_B$ and $r > r_B$, respectively. In the region I, a coincident brane (of the SM- and the background NS-branes) is placed at $\vec{r} = 0$, and we want this to be a codimension-2 brane (see section 8). In the region II, we want a corresponding brane to be placed at $r = 2r_B (\equiv r_m)$ by reason of symmetry. So the configuration is that we have a codimension-2 brane at $\vec{r} = 0$, and a corresponding brane at $r = r_m$ (which can be either the type II codimension-1 brane, or a codimension-2 brane), and finally a type I codimension-1 brane (= brane B) in the middle, i.e., at $r = r_B$. The whole internal space Σ_2 thus consists of two parts; i.e., the region I ($\equiv 0 < r < r_B \equiv \Sigma_{2I}$) and the region II ($\equiv r_B < r < r_m \equiv \Sigma_{2II}$), where $r_m = 2r_B$ as defined above.

In the region I, $\psi(r)$ and $i_M(r)$ are due to the brane at $\vec{r} = 0$ and directly given by (4.9) (or (4.11)) and (4.12), respectively. In the region II, $\psi(r)$ and $i_M(r)$ are due to the brane

at $r = r_m$ and also given by (4.9) (or (4.11)) and (4.12), but this time r is replaced with $r_m - r$. That is, we construct Σ_2 by gluing Σ_{2II} onto Σ_{2I} with left and right reversed. We have

$$\psi(r) = \begin{cases} b_{0I} r^{-2} [c_{1I} (r/r_{0I})^{\gamma_I} \pm c_{2I} (r/r_{0I})^{-\gamma_I}]^{-2} & (\text{region I}) \\ b_{0II} r^{-2} [c_{1II} ((r_m - r)/r_{0II})^{\gamma_{II}} \pm c_{2II} ((r_m - r)/r_{0II})^{-\gamma_{II}}]^{-2} & (\text{region II}), \end{cases} \quad (9.5)$$

and

$$i_M(r) = \begin{cases} (r/\tilde{r}_{0I})^{\alpha_{MI}} & (\text{region I}) \\ ((r_m - r)/\tilde{r}_{0II})^{\alpha_{MII}} & (\text{region II}), \end{cases} \quad (9.6)$$

where c_i are given by $c_1 = c_2 = 1$ for $\Lambda \neq 0$, and $c_1 = 0, c_2 = 1$ for $\Lambda = 0$. Also, the constant b_0 is $b_0 = \pm(8\gamma^2/m\Lambda)$ for $\Lambda \neq 0$, and arbitrary for $\Lambda = 0$. Substituting (9.5) and (9.6) into (9.1)–(9.3) gives

$$\alpha_{RI} + \alpha_{RII} = \mathcal{B}_1, \quad \alpha_{\Phi I} + \alpha_{\Phi II} = -\frac{(p+1)}{2}\mathcal{B}_2 - \frac{1}{2}\mathcal{B}_1, \quad \alpha_{BI} + \alpha_{BII} = -2\mathcal{B}_2, \quad (9.7)$$

where \mathcal{B}_1 and \mathcal{B}_2 are given by

$$\mathcal{B}_1 = \kappa^2 C_1^{(p+1)} + \frac{2}{m} Y_0, \quad \mathcal{B}_2 = \kappa^2 C_2^{(p+1)} + \frac{(\alpha+2)}{2m} Y_0, \quad (9.8)$$

where

$$Y_0 \equiv \gamma_I \frac{[c_{1I} X_I \mp c_{2I} X_I^{-1}]}{[c_{1I} X_I \pm c_{2I} X_I^{-1}]} + \gamma_{II} \frac{[c_{1II} X_{II} \mp c_{2II} X_{II}^{-1}]}{[c_{1II} X_{II} \pm c_{2II} X_{II}^{-1}]}, \quad \left(X_{I,II} \equiv \left(\frac{r_B}{r_{0I,II}} \right)^{\gamma_{I,II}} \right). \quad (9.9)$$

Recalling (4.16), one finds that (9.7) can be solved by setting

$$\mathcal{B}_1 = \mathcal{B}_2 = 0, \quad (9.10)$$

and therefore

$$\alpha_{MI} + \alpha_{MII} = 0, \quad (M \equiv R, \Phi, B). \quad (9.11)$$

With $\Psi_i = 0$, (9.11) reduces to a fine-tuning of $V_p(\Phi)$. But in the case $\Lambda = 0$, once Ψ_i is "turned on" (9.11) does not restrict $V_p(\Phi)$ anymore. The reason is as follows. In (9.11), the number of independent equations is only two (instead of three) because they are related with each other by (4.16). The number of independent α_M is also two because they depend only on two independent functions $V_p(\Phi)$ and $\partial V_p(\Phi)/\partial\Phi$ through C_1 and C_2 ; i.e., four α_M contain only two independent degrees of freedom (indeed, four α_M with $M \equiv R, \Phi, B, \xi$ are related with each other by (4.15), and (4.16) as well). So the two degrees of freedom $V_p(\Phi)$ and $\partial V_p(\Phi)/\partial\Phi$ are fixed by the equations in (9.11). If Ψ_i is "turned on", however, the situation changes. C_1 and C_2 (and therefore α_M) are now going to include two more degrees of freedom F_Φ and F_B (see (3.26) and (5.2)); i.e., α_M become

$\alpha_M = \alpha_M(V_p(\Phi), \partial V_p(\Phi)/\partial\Phi, F_\Phi, F_B)$. Thus this time the two independent equations in (9.11) can not restrict $V_p(\Phi)$ anymore⁵ due to the presence of the extra degrees of freedom F_Φ and F_B , and therefore (9.11) does not reduce to a fine-tuning of $V_p(\Phi)$ when $\Psi_i \neq 0$. In the case $\Lambda \neq 0$, however, the self-tuning of $V_p(\Phi)$ is not guaranteed since in this case we have only one extra degree of freedom, i.e., F .

Turning back to the matching conditions, it is convenient to replace the conditions in (9.4) by

$$\Psi_I|_{r=r_B} = \Psi_{II}|_{r=r_B}, \quad (9.12)$$

and

$$i_{MI}|_{r=r_B} = i_{MII}|_{r=r_B}. \quad (9.13)$$

The condition (9.13) can be easily satisfied if we take

$$\tilde{r}_{0I} = \tilde{r}_{0II} = r_B. \quad (9.14)$$

But (9.12) gives a condition

$$b_{0I} [c_{1I} X_I \pm c_{2I} X_I^{-1}] = b_{0II} [c_{1II} X_{II} \pm c_{2II} X_{II}^{-1}], \quad (9.15)$$

by which, (9.9) reduces to

$$Y_0 = (\gamma_I + \gamma_{II}) \frac{[c_{1I} X_I \mp c_{2I} X_I^{-1}]}{[c_{1I} X_I \pm c_{2I} X_I^{-1}]}. \quad (9.16)$$

Also from (4.10) one finds that (9.11) implies

$$\gamma_I + \gamma_{II} = 0 \quad \rightarrow \quad Y_0 = 0 \quad (9.17)$$

provided m (or α) takes the same value at both regions of Σ_2 , and (9.17) in turn implies (by (9.8) and (9.10)) that

$$C_1^{(p+1)} = C_2^{(p+1)} = 0. \quad (9.18)$$

This is interesting. If Λ couples with the dilaton in the same form at both regions of Σ_2 (but see also the case I of the next section), we do not need to introduce the brane B in order to satisfy the matching conditions at $r = r_B$. They are automatically satisfied as long as (9.11) and (9.15) are met.

⁵In this case, (9.11) only determines the boundary values of F_Φ and F_B , or equivalently the values of Ψ_i at $r = 0$ and $r = r_m$. Ψ_i have been introduced in the action in order to compensate the D-brane tensions at the boundaries $r = 0$ and $r = r_m$, where the D-branes are assumed to be located. So, what really matters is just the boundary values of Ψ_i at $r = 0$ and $r = r_m$, not the functional form of Ψ_i between $r = 0$ and $r = r_m$. Indeed, the equations of motion for Ψ_i are second order equations, i.e., $\square\Psi_i = 0$ with \hat{g}_{ab} in \square replaced by $e^{4\Phi}\hat{g}_{ab}$ as can be obtained from (2.7). So the boundary values of Ψ_i at both $r = 0$ and $r = r_m$ can be chosen to compensate the D-brane tensions as we wish.

10. Brane world models

In this section we will consider various types of brane world models satisfying the matching conditions of the previous section. In the followings we will restrict our discussion only to the case where $|\gamma_I| = |\gamma_{II}|$ in consideration of symmetry. Also we will assume that the value of Λ of the region I ($\equiv \Lambda_I$) can be different from that of the region II ($\equiv \Lambda_{II}$).

(a) case I ($\Lambda_I \neq 0, \Lambda_{II} \neq 0$)

Since γ must be positive in the case $\Lambda \neq 0$, γ_I and γ_{II} are both positive constants, i.e., $\gamma_I = \gamma_{II}$, which agrees with (9.11) only if $m_I = -m_{II}$, where $m_{I,II}$ represent the values of m at region I and region II, respectively. Having different m at each region means that Λ couples with dilaton differently at each region. As an example, consider a case where $\alpha = -2$ ($m = 2$) in the region I, while $\alpha = 0$ ($m = -2$ assuming that $p = 3$) in the region II. This describes a model in which Λ couples with dilaton with a factor $e^{-2\Phi}$ in the region I, while it does not couple with dilaton in the region II. Since $m_I = 2$ and $m_{II} = -2$, $\gamma_{I,II}$ become $\gamma_I = -\alpha_{\xi I}$ and $\gamma_{II} = \alpha_{\xi II}$, respectively, and (9.11) implies $\gamma_I = \gamma_{II}$.

The analysis of section 9 was based on the assumption that $\alpha_I = \alpha_{II} = \alpha$ ($m_I = m_{II} = m$). If $\alpha_I \neq \alpha_{II}$ ($m_I \neq m_{II}$), \mathcal{B}_1 and \mathcal{B}_2 in (9.8) must be modified. Upon using (9.15), they are modified to

$$\mathcal{B}_1 = \kappa^2 C_1^{(p+1)} + 2 \left(\frac{\gamma_I}{m_I} + \frac{\gamma_{II}}{m_{II}} \right) \frac{[c_{1I} X_I \mp c_{2I} X_I^{-1}]}{[c_{1I} X_I \pm c_{2I} X_I^{-1}]}, \quad (10.1)$$

$$\mathcal{B}_2 = \kappa^2 C_2^{(p+1)} + \frac{1}{2} \left[(\alpha_I + 2) \frac{\gamma_I}{m_I} + (\alpha_{II} + 2) \frac{\gamma_{II}}{m_{II}} \right] \frac{[c_{1I} X_I \mp c_{2I} X_I^{-1}]}{[c_{1I} X_I \pm c_{2I} X_I^{-1}]}. \quad (10.2)$$

In the case $m_I = -m_{II}$ (and $\gamma_I = \gamma_{II}$), (10.1) reduces to $\mathcal{B}_1 = \kappa^2 C_1^{(p+1)}$, and consequently $\mathcal{B}_1 = 0$ implies

$$C_1^{(p+1)} = 0 \quad \rightarrow \quad V_{p+1}(\Phi) = 0. \quad (10.3)$$

So the brane B is unnecessary even in this case. Since (10.3) also implies $C_2^{(p+1)} = 0$, the condition $\mathcal{B}_2 = 0$ requires

$$\left[\frac{(\alpha_I + 2)}{m_I} + \frac{(\alpha_{II} + 2)}{m_{II}} \right] (X_I \mp X_I^{-1}) = 0, \quad (10.4)$$

where we have set $\gamma_I = \gamma_{II}$ and $c_{1I} = c_{2I} = 1$. For $\alpha_I = -2$ ($m_I = 2$) and $\alpha_{II} = 0$ ($m_{II} = -2$), (10.4) is satisfied only when $X_I = X_I^{-1}$ (and therefore when $\Lambda > 0$), or equivalently when

$$r_{0I} = r_{0II} = r_B, \quad (10.5)$$

which together with (9.14) implies (see the definition of d_M in (6.4))

$$d_{MI} = d_{MII} = 1. \quad (10.6)$$

Thus in this case the integration constants \tilde{r}_0 and r_0 (and therefore d_M) are all fixed by the matching conditions. Finally, the geometry of the bulk spacetime is described by (6.10) (or (6.14) in the case $\alpha = -2$) at both regions of Σ_2 .

(b) case II ($\Lambda_I \neq 0, \Lambda_{II} = 0$)

In the case II (and in the case III) m_I and m_{II} do not have to be different from each other, and in the followings we will restrict our discussion only to the case $m_I = m_{II} = 2$. Since $m_I = m_{II}$, the whole discussion of section 9 can be applied to the case II (and the case III). The brane B is unnecessary, and the matching conditions to be met are those in (9.11), (9.14) and (9.15). Since $m_I = m_{II}$, (9.11) implies $\gamma_{II} = -\gamma_I < 0$, which is allowed only if $\Lambda_{II} = 0$. (9.15) therefore reduces to

$$b_{0I} [X_I \pm X_I^{-1}] = b_{0II} X_{II}^{-1}, \quad (10.7)$$

where b_{0I} is given by $b_{0I} = \pm(4\gamma_I^2/\Lambda_I)$, but b_{0II} is arbitrary. So (10.7) can be satisfied for any $r_{0I,II}$ if b_{0II} is chosen properly. Since $r_{0I,II}$ are arbitrary, $d_{MI,II}$ are not fixed by the definition of d_M in (6.4). The geometry of the bulk spacetime is given by (6.14) in the region I, while it is given by (6.15) in the region II.

(c) case III ($\Lambda_I = \Lambda_{II} = 0$)

In the case III, we only need to consider the case $m_I = m_{II} = 2$ (recall that we have decided to take $\alpha = -2$ ($m = 2$) in the case $\Lambda = 0$, see section 6). So the whole discussion of section 9 can be applied to the case III either; the brane B is unnecessary, and the matching conditions to be met are just (9.11), (9.14) and (9.15) as before. But in the case III, (9.15) reduces to

$$\frac{b_{0I}}{X_I} = \frac{b_{0II}}{X_{II}}, \quad (10.8)$$

where b_{0I} and b_{0II} are both arbitrary. Since $b_{0I,II}$ are arbitrary, (10.8) can be satisfied for any arbitrary $r_{0I,II}$, and therefore $d_{MI,II}$ are not fixed by the matching conditions. (9.11), on the other hand, implies that $\gamma_I = -\gamma_{II} > 0$, so Σ_2 is closed at $\eta = 0$ ($r = 0$) (note that the bulk geometry of the case III is given by (6.15)), while it diverges at $\eta = \eta_m$ ($r = r_m$). Thus the brane at $\eta = 0$ is a codimension-2 brane, while the one at $\eta = \eta_m$ is necessarily a codimension-1 brane. If we identify these two branes as the $\eta_c \rightarrow 0$ limits of the type II codimension-1 brane, they become T-duals⁶ of each other because they have the same mass (note that $|\gamma_I| = |\gamma_{II}|$ in the above).

So far we have considered three types of brane world models satisfying the matching conditions, and in all these three cases the brane B at $r = r_B$ is unnecessary. Among these, the case I and case II are somewhat special in the sense that Λ or its dilaton coupling is not uniform in the whole regions of Σ_2 . Besides this, the self-tuning of $V_p(\Phi)$ is not obvious in the case I due to lack of extra degrees of freedom which is needed to avoid the fine-tuning of $V_p(\Phi)$ (see section 9). (Also in the case I, (10.6) is not consistent with $d_\Phi \rightarrow 0$.) Of the most interest is the case III, which is also natural in the context of string theory where Λ is absent in ordinary circumstances.

⁶To be precise, they are "anti T-duals" (rather than T-duals) of each other in the sense that their masses are equal in magnitude, but opposite in sign (see (9.11)).

In the next we will be back to the case III to go into more details about the brane world model with $\Lambda = 0$.

(d) case III again

The self-tuning brane world model with $\Lambda = 0$ includes two coincident branes, one of which is a codimension-2 brane placed at $\vec{\eta} = 0$ (the origin of Σ_2), another a codimension-1 brane placed at $\eta = \eta_m$ (the edge of Σ_2). These two branes are (anti) T-duals of each other, and related by the duality relation (8.3). So the codimension-2 brane in the η -coordinates becomes a codimension-1 brane in the $\tilde{\eta}$ -coordinates, and *vice versa*. Namely, these two branes interchange their shapes under (8.3), and one of them is identified with our SM-brane (plus the background NS-brane).

Let us turn to the geometry of Σ_2 especially in the vicinity of $\vec{\eta} = 0$. Since the bulk geometry is practically unaffected by the SM-brane (see section 7), we simply consider the case where $V_D(\Phi)$ is "turned off" and there is only a background NS-brane at $\vec{\eta} = 0$. With $V_D(\Phi)$ turned off ($l = 0$), (8.1) can be written as

$$ds_2^2 = d\eta^2 + \beta^2 \eta^2 d\theta^2 + (-dt^2 + d\vec{x}_p^2), \quad \left(\beta \equiv \frac{1}{d_\xi} \frac{r_0}{\eta_0} \right), \quad (10.9)$$

where β is a dimensionless constant associated with a deficit angle δ defined by $\delta = 2\pi(1 - \beta)$. According to the string theoretical description (6.19), β becomes $\beta = g_s^2 \alpha_\xi$ and further, since $\alpha_\xi = \kappa^2 V_p(\Phi)/2\pi$ (note that $V_p(\Phi) = (-1/2)\partial V_p(\Phi)/\partial\Phi$ for $l = 0$), and $V_p(\Phi) (= V_{\text{NS}}(\Phi)) \sim 1/\alpha'^{(p+1)/2} g_s^2$ in the string theory, β finally becomes

$$\beta \sim \left(\frac{M_s}{M_{3+p}} \right)^{1+p}, \quad (10.10)$$

where M_s is the string scale: $M_s = 1/l_s = 1/\sqrt{\alpha'}$. If $\beta = 1$, Σ_2 is simply R_2 . But if $\beta = 1/n$, Σ_2 becomes an orbifold R_2/Z_n with an orbifold singularity at $\vec{\eta} = 0$. But in both cases β should be of order one; $\beta \sim O(1)$, which is naturally connected with the hierarchy problem. Namely in (10.10), M_s and M_{3+p} should be of the same order in order that $\beta \sim O(1)$, which accords with an assumption [1–3] that there exists only one fundamental short distance scale (i.e., the electroweak scale m_{EW}) in nature.

11. 4d Planck scale

In this section we will restrict our discussion to the case $p = 3$ to evaluate the 4d Planck scale M_{pl} . The finiteness of M_{pl} is closely related with the localization of the zero mode of the 4d graviton. We will also concentrate our attention mostly on the case III among the three cases of section 10. Finally we will use $l \cong 0$ by the same reason that was used to obtain (10.10).

In the case III, the 6d metric is given by (8.1) in the region I ($0 \leq \eta \leq \eta_B$), while it is given by (8.2) with η replaced by $\eta_m - \eta$ in the region II ($\eta_B < \eta \leq \eta_m$) (also η_0 must be

identified with η_B in order that two metrics match each other at $\eta = \eta_B$). For this metric, M_{pl} is given by

$$M_{\text{pl}}^2 = 2\pi M_6^4 \frac{|\alpha_\xi|}{d_B} \left(\int_\tau^{\eta_B} \eta d\eta + \eta_B^4 \int_{\eta_B}^{\eta_m - \tau} \frac{d\eta}{(\eta_m - \eta)^3} \right), \quad (11.1)$$

where we have used (6.17) (together with (6.13)) and the relation $d_\xi = d_\Phi^{-2} d_B^2$. In (11.1), τ represents the thickness (we assume that every brane has the same thickness) of the branes in the η -coordinates. In the thin brane limit, τ vanishes. But in reality, branes have nonzero thickness and τ takes some nonzero value. Neglecting τ^2 term, one finds

$$M_{\text{pl}}^2 = 2\pi M_6^4 \frac{|\alpha_\xi|}{d_B} \rho_{\text{max}}^2, \quad \left(\rho_{\text{max}} \equiv \frac{\eta_B^2}{\tau} \right), \quad (11.2)$$

where ρ_{max} may be identified with the size of the codimension-1 brane, or equivalently the size of Σ_2 ; indeed, if the codimension-2 brane has a size (thickness) τ , then the codimension-1 brane must have a size $\eta_B^2/\tau (= \rho_{\text{max}})$ by (8.3) because they are (anti) T-duals of each other. Since $\alpha_\xi = \kappa^2 V_p(\Phi)/2\pi$ for $l = 0$, (11.2) finally becomes

$$M_{\text{pl}}^2 \sim |V_p(\Phi)| \rho_{\text{max}}^2, \quad (11.3)$$

where we have set $d_B = 1$ (note that $a_B = 0$ for $l = 0$ and $\alpha = -2$). (11.3) contrasts with the conventional equation [2]

$$M_{\text{pl}}^2 \sim M_6^4 \rho_{\text{max}}^2. \quad (11.4)$$

In (11.4), the 4d Planck scale M_{pl} is given in terms of the 6d Planck scale M_6 . But in (11.3), M_{pl} is not directly given by M_6 (M_6^4 was cancelled out in (11.3)); it is determined by the brane tensions and the size of the extra dimensions. But if we use $V_p(\Phi) \sim V_{\text{NS}}(\Phi) \sim 1/\alpha'^2 g_s^2$, (11.3) can be written in terms of M_s :

$$M_{\text{pl}}^2 \sim \frac{M_s^4}{g_s^2} \rho_{\text{max}}^2, \quad (11.5)$$

where M_s is the string scale: $M_s = 1/l_s = 1/\sqrt{\alpha'}$ as mentioned before. In some sense, (11.5) may be considered as a string theoretic generalization of (11.4); (11.5) reduces to (11.4) if we identify

$$M_6^4 = \frac{M_s^4}{g_s^2}, \quad (11.6)$$

which is just an analogue of the string theoretical definition of the 10d Planck scale: $M_{10}^8 = M_s^8/g_s^2$. In the limit $g_s \rightarrow 0$, however, (11.6) is unnatural because it implies $M_6/M_s \rightarrow \infty$ as $g_s \rightarrow 0$. (11.6) may be sensible perhaps when $g_s \sim O(1)$. That is, the identification of (11.5) with (11.4) may not be valid in the weakly coupled case $g_s \rightarrow 0$.

There is a different way of viewing (11.5). If we take an assumption that m_{EW} is the only fundamental short distance scale in nature (i.e., if we assume that $M_s \sim TeV$ and $\rho_{\text{max}} \sim TeV^{-1}$), then g_s expected from (11.5) would be $\sim 10^{-16}$, which is just the realistic decoupling limit considered in "Little String Theories at a TeV " [20].

Finally, M_{pl} of the case II (of section 10) is similar to that of the case III. For $m_I = m_{\text{II}} = 2$ ($\alpha_I = \alpha_{\text{II}} = -2$), it is given by

$$M_{\text{pl}}^2 = 2\pi M_6^4 \frac{|\alpha_\xi|}{d_B} \left(\frac{1}{\sqrt{|\Lambda|}} \int_\tau^{\eta_B} \sinh \sqrt{|\Lambda|} \eta d\eta + \eta_B^4 \int_{\eta_B}^{\eta_m - \tau} \frac{d\eta}{(\eta_m - \eta)^3} \right). \quad (11.7)$$

Note that (11.7) reduces to (11.1) in the limit $\Lambda \rightarrow 0$. For the case I, we will not give a precise value of M_{pl} , but it is obviously finite even in the thin brane limit $\tau \rightarrow 0$. Omitting η_B -dependent hyperbolic functions, it is of order $M_{\text{pl}}^2 \sim |V_p(\Phi)|/|\Lambda|$, or in terms M_s it is given by $M_{\text{pl}}^2 \sim M_s^4/|\Lambda|g_s^2$. But as mentioned before, the self-tuning is not guaranteed in this case.

12. Summary

In this paper we have presented a new type of self-tuning mechanism for $(3+p)$ d brane world models in the framework of gravity-scalar theory. Each model contains two coincident branes each of which is a stack of a D-brane (SM-brane) and a background NS-brane. Among these models, of the most interest is the case with $\Lambda = 0$, which is not only interesting but also natural in the context of the string theory. In this model, one of the coincident branes is a codimension-2 brane placed at the origin $\vec{\eta} = 0$ of the 2d transverse space Σ_2 , while the other is a codimension-1 brane placed at the edge of Σ_2 . These two branes are (anti) T-duals of each other, and interchange their shapes under duality transformation, and one of them is identified as our SM-brane (plus the background NS-brane).

The given models exhibit a remarkable feature. In the limit $g_s \rightarrow 0$, the bulk geometry (as well as the flat intrinsic geometry of the branes) is practically insensitive to the quantum fluctuations of SM-fields with support on the SM-brane. This can be achieved by introducing NS-brane which serves as a background brane on which our SM-brane is to be set. Indeed, the existence of the background NS-brane is naturally suggested by field equations, which impose a certain restriction on (the dilaton coupling of) the brane tension so that the background brane must be of the NS-NS type. In the presence of this NS-brane the 2d transverse space Σ_2 becomes an orbifold R_2/Z_n with a deficit angle $\delta = 2\pi(1 - \beta)$ where $\beta \sim (M_s/M_{3+p})^{1+p}$. So in order that β becomes of order one, the $(3+p)$ d Planck scale M_{3+p} should be of the same order as the string scale M_s , which accords with the hierarchy conjecture that there may exist only one fundamental short distance scale in nature.

Now introduce the SM-brane on top of the background NS-brane placed at the orbifold singularity. Such an introduction of an SM-brane usually affects the geometry of bulk spacetime due to the tension of the SM-brane. In our case the effect of the (brane tension of the) SM-brane on the bulk geometry is essentially expressed in terms of the parameters l and k . So they take nonzero values in the presence of the SM-brane, while they vanish in the absence of the SM-brane. In the given models, however, the increments in l and k due to an introduction of the SM-brane is negligibly small; they are all of the order $\sim g_s$, so they go to zero in the limit $g_s \rightarrow 0$. The shifts in l and k due to quantum corrections to

the brane tension (due to dynamics of SM fields on the SM-brane) are consequently even smaller than this. They are just $\delta l, \delta k \sim g_{YM}^2 g_s \alpha'^{-(p-3)/2}$ (or $\delta l, \delta k \sim g_{YM}^4 \alpha'^{-(p-3)} \sim g_s^2$ in the case $g_{YM}^2 \sim g_s \alpha'^{-(p-3)/2}$), which again vanish in the limit $g_s \rightarrow 0$. Thus the geometry of the bulk spacetime is practically insensitive to the quantum fluctuations of SM fields with support on the SM-brane.

Finally for $p = 3$, the 4d Planck scale M_{pl} is given (in the case $\Lambda = 0$) by $M_{\text{pl}}^2 \sim M_s^4 \rho_{\text{max}}^2 / g_s^2$. So if we apply to this equation the hierarchy assumption that M_s and ρ_{max} are both of the order the electroweak scale $\sim TeV$, the estimated value for g_s will be about $\sim 10^{-16}$, which, however, is just the realistic value of the decoupling limit $g_s \rightarrow 0$. This again shows that taking $g_s \rightarrow 0$ is consistent with the hierarchy conjecture in the present paper.

Acknowledgments

This work was supported by Kyungsung University in 2005.

A. Codimension-1 branes

In this section we will consider two different types of codimension-1 branes, which we will call type I or type II codimension-1 brane, respectively.

A. Type I

Type I codimension-1 brane (this is an ordinary codimension-1 brane) is described by

$$I_{\text{cod-1}}^{(I)} = - \int d^{p+1} x d\theta \sqrt{-\det |g_{\mu\nu}|} \sqrt{g_{\theta\theta}} V_{p+1}(\Phi) , \quad (\text{A.1})$$

where $g_{\theta\theta}$ (as well as $g_{\mu\nu}$) is a pullback of $G_{\theta\theta}$ to the codimension-1 brane. Upon using (2.2) and (2.6), (A.1) reduces to

$$I_{\text{cod-1}}^{(I)} = - \int_{\Sigma_3} d^3 y \sqrt{\hat{g}_{tt}} \sqrt{\hat{g}_{\theta\theta}} e^{4\Phi - \frac{p}{2}B} V_{p+1}(\Phi) \delta(r - r_B) , \quad (\text{A.2})$$

where r_B represents the position of the codimension-1 brane. With this action, the field equations become

$$\frac{d}{dr} \left(r \frac{d \ln R}{dr} \right) + \Lambda \psi r = -\kappa^2 C_1^{(p+1)} \delta(r - r_B) , \quad (\text{A.3})$$

$$\begin{aligned} \frac{d}{dr} \left(r \frac{d\Phi}{dr} \right) - \frac{[(\alpha + 2)(p + 1) + 4]}{8} \Lambda \psi r &= \frac{(p + 1)}{2} \kappa^2 C_2^{(p+1)} \delta(r - r_B) \\ &+ \frac{1}{2} \kappa^2 C_1^{(p+1)} \delta(r - r_B) , \end{aligned} \quad (\text{A.4})$$

$$\frac{d}{dr} \left(r \frac{dB}{dr} \right) - \frac{(\alpha + 2)}{2} \Lambda \psi r = 2\kappa^2 C_2^{(p+1)} \delta(r - r_B) , \quad (\text{A.5})$$

where the constants $C_i^{(p+1)}$ are defined by

$$\begin{aligned} C_1^{(p+1)} &= e^{(p+1)B/2} \sqrt{g_{\theta\theta}} V_{p+1}(\Phi) \Big|_{r=r_B}, \\ C_2^{(p+1)} &= e^{(p+1)B/2} \sqrt{g_{\theta\theta}} \left(V_{p+1}(\Phi) + \frac{1}{2} \frac{\partial V_{p+1}(\Phi)}{\partial \Phi} \right) \Big|_{r=r_B}. \end{aligned} \quad (\text{A.6})$$

B. Type II

Type II codimension-1 brane can be obtained from (A.1) (or (A.2)) by setting

$$\sqrt{g_{\theta\theta}} V_{p+1}(\Phi) \equiv \frac{V_p(\Phi)}{2\pi}. \quad (\text{A.7})$$

So the action is given by

$$I_{\text{cod-1}}^{(II)} = - \int_{\Sigma_3} d^3y \sqrt{-\hat{g}_{tt}} e^{2\Phi} V_p(\Phi) \frac{\delta(r)}{2\pi}, \quad (\text{A.8})$$

where we have assumed that Σ_2 is not closed at $r = 0$ and the brane is placed there. (A.7) indicates that $V_{p+1}(\Phi)$ of the type II brane is inversely proportional to the size (the radius) of the brane; i.e., $V_{p+1}(\Phi) \propto 1/\sqrt{g_{\theta\theta}}$ so that $V_p(\Phi)$ becomes independent of $\sqrt{g_{\theta\theta}}$. This is the crucial difference between type I and Type II codimension-1 branes. In the case of type I brane, $V_{p+1}(\Phi)$ is itself independent of $\sqrt{g_{\theta\theta}}$. (A.8) has the same form as (3.3) except that the 2d delta-function $\delta^2(\vec{r})$ is replaced by 1d delta-function $\delta(r)/2\pi \sqrt{g_2}$. The type II codimension-1 brane is a codimension-1 brane which can be obtained from a codimension-2 brane (a point) by expanding it to a circle with the total mass of the brane kept constant. Conversely, a codimension-2 brane can be obtained from the type II codimension-1 brane by shrinking it to a point with the total mass of the brane kept constant. (A.7) indicates the fact that the total mass of the brane is unchanged under this expansion (or contraction). The field equations for this codimension-1 brane are given by

$$\frac{d}{dr} \left(|r| \frac{d \ln R}{dr} \right) + \Lambda \psi |r| = -2\kappa^2 C_1 \frac{\delta(r)}{2\pi}, \quad (\text{A.9})$$

$$\frac{d}{dr} \left(|r| \frac{d\Phi}{dr} \right) - \frac{[(\alpha+2)(p+1)+4]}{8} \Lambda \psi |r| = (p+1)\kappa^2 C_2 \frac{\delta(r)}{2\pi}, \quad (\text{A.10})$$

$$\frac{d}{dr} \left(|r| \frac{dB}{dr} \right) - \frac{(\alpha+2)}{2} \Lambda \psi |r| = 4\kappa^2 C_2 \frac{\delta(r)}{2\pi}, \quad (\text{A.11})$$

where the constants C_i are defined by

$$C_1 = e^{(p+1)B/2} V_p(\Phi) \Big|_{r=0}, \quad C_2 = e^{(p+1)B/2} \left(V_p(\Phi) + \frac{1}{2} \frac{\partial V_p(\Phi)}{\partial \Phi} \right) \Big|_{r=0}, \quad (\text{A.12})$$

similarly to (3.26). Note that the right hand sides of (A.9)–(A.11) have been doubled as compared with (3.21)–(3.23). The reason is as follows. The left hand sides of (A.9)–(A.11) show that there is a reflection symmetry about the codimension-1 brane at $r = 0$ (we have $|r|$ in the equations). So we naturally take the codimension-1 brane

as an orbifold fixed line: i.e., we identify every point of the region $r < 0$ with the corresponding point of the region $r > 0$, and then take the region $r > 0$ as a fundamental domain. Then the orbifold fixed line at $r = 0$ becomes a stack of two codimension-1 branes each of which belongs to the corresponding regions. So we have to double the right hand sides of (A.9)–(A.11). The equations (A.9)–(A.11) can be solved by (4.1) with $\psi(r)$ and $i_M(r)$ given by

$$\frac{d}{dr} \left(|r| \frac{d \ln \psi r^2}{dr} \right) + \frac{m\Lambda}{|r|} \psi r^2 = -2m\alpha_\xi \delta(r) \quad (\text{A.13})$$

and

$$\frac{d}{dr} \left(|r| \frac{d \ln \hat{i}_M}{dr} \right) = 2\alpha_M \delta(r) , \quad (\hat{i}_M \equiv i_M r^{-2k_M}) \quad (\text{A.14})$$

where α_M are the same ones as those in (4.4)–(4.7).

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